Mean–Semivariance Portfolio Selection: Single Period vs Continuous Time

Xun Yu Zhou

The Chinese University of Hong Kong

July 5/ Cambridge

Markowitz's (Original) Model

- Single period
- $m(\geq 2)$ securities, each with total return R_j, $j = 1, \cdots, m$
- $r_j = E[R_j]$, $\sigma_{ij} = \text{Cov}\ (R_i, R_j)$
- An agent with fund x_0, and a targeted expected payoff z at the end of the period
- To find a portfolio $\pi = (\pi_1, \cdots, \pi_m)$ so as to

$$
\begin{array}{ll}
\text{Minimize} & \sum_{i,j=1}^{m} \sigma_{ij}\pi_i\pi_j \ \ \text{(risk)} \\
\text{subject to} & \sum_{i=1}^{m} \pi_i = x_0 \ \ \text{(budget constraints)} \\
 & \sum_{i=1}^{m} r_i\pi_i = z \ \ \text{(targeted payoff)} \\
 & [\ \ \pi_i \geq 0, \ \ i = 1, \cdots, m \ \ \text{(no shorting)} \ \ ]
\end{array}
$$

- The optimal portfolio is called an *efficient portfolio* corresponding to z

Risk Measures

- *Risk*: chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
- *Variance/covariance* used to measure risk by Markowitz (1952)
- Criticisms on using variance include
  - penalty on upside return
  - equal weight on upside and downside whereas asset return distribution generally assymmetric
- *Semivariance* proposed where only the return below its mean or a target level counted as risk (Markowitz 1959: "*semivariance seems more plausible than variance as a measure of risk*")
- Generalization of semivariance: *Downside risk* (Fishburn 1977, Sortino and van der Meer 1991)
- Other risk measures: lower partial moment (Bawa 1975), mean–absolute deviation (Konno and Yamazaki 1991), minimax measure (Cai, Teo, Yang and Z. 2000) ...

A Mean–Semivariance Model

- Single period
- $m(\geq 2)$ securities, each with total return R_j, $j = 1, \cdots, m$
- $r_j = E[R_j]$, $E[R_j^2] < \infty$
- An agent with fund x_0, and a targeted expected payoff z at the end of the period
- To find a portfolio $\pi = (\pi_1, \cdots, \pi_m)$ so as to

$$
\begin{array}{ll}
\text{Minimize} & E\left[\left(\sum_{i=1}^m \pi_i R_i - \sum_{i=1}^m \pi_i r_i\right)_-\right]^2 \\
\text{subject to} & \sum_{i=1}^m \pi_i = x_0 \\
 & \sum_{i=1}^m r_i \pi_i = z
\end{array}
\tag{1}
$$

A Continuous-Time Market

- A market in which $m+1$ *securities* (*assets*) traded continuously
- Market randomness described by a complete filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ P)$ along with an $\mathbb{R}^m$-valued, $\mathcal{F}_t$-adapted standard Brownian motion $W(t) = (W^1(t), \cdots, W^m(t))'$ with $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $W(\cdot)$
- A *bond* (or a *bank account*) whose price process $S_0(t)$ satisfies

$$dS_0(t) = r(t)S_0(t)dt,\ t \in [0,T];\ S_0(0) = s_0 > 0,$$

where $r(t) > 0$: *interest rate*

- m *stocks* whose price processes $S_1(t), \cdots S_m(t)$ satisfy stochastic differential equation (SDE)

$$\begin{cases} dS_i(t) = S_i(t)\big\{\mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t)\big\}, \\ \qquad\qquad t \in [0,T]; \\ S_i(0) = s_i > 0, \end{cases}$$

where $\mu_i(t)$: *appreciate rate*, $\sigma_{ij}(t)$: *volatility* (*dispersion*) *rate*

Wealth Equation

- $r(t), \mu_i(t), \sigma_{ij}(t)$ all uniformly bounded, $\mathcal{F}_t$-adapted stochastic processes
- Define *covariance matrix*

$$\sigma(t) := (\sigma_{ij}(t))_{m\times m}$$

- Assume: $\sigma(t)\sigma(t)' \geq \delta I$ for some $\delta > 0$
- An agent's (self-financed) *wealth process* $x(t)$ follows *wealth equation*

$$\begin{cases} dx(t) &= \Big\{ r(t)x(t) + \sum_{i=1}^m \bar{\mu}_i(t)\pi_i(t) \Big\} dt \\ & \quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)\pi_i(t) dW^j(t), t \in [0,T]; \\ x(0) &= x_0, \end{cases}$$

where $\bar{\mu}_i(t) := \mu_i(t) - r(t)$, and $\pi_i(t)$ the allocation of wealth (in dollar amount) in the ith stock

Admissible Portfolios

- A *portfolio*
$$\pi(t) = (\pi_1(t), \cdots, \pi_m(t))'$$

- A portfolio $\pi(\cdot)$ called *admissible* if $\pi(\cdot)$ is $\mathcal{F}_t$-adapted and $E\int_0^T |\pi(t)|^2 dt < +\infty$

- Set

  - $B(t) := (\mu_1(t) - r(t), \cdots, \mu_m(t) - r(t))$ (*excess rate of return process*)

  - $\theta(t) \equiv (\theta_1(\cdot), \cdots, \theta_m(\cdot)) := B(t)(\sigma(t)')^{-1}$ (*risk premium process*)

  - Wealth equation in matrix form

$$\begin{cases} dx(t) & = \{r(t)x(t) + B(t)\pi(t)\}dt + \pi(t)'\sigma(t)dW(t), \\ x(0) & = x_0 \end{cases}$$

Mean–Risk Model

Mean–risk portfolio selection problem is formulated as the following optimization problem parameterized by a pair of scalars (x_0, z):

$$\text{Minimize} \quad J_{(x_0,z)}(\pi(\cdot)) = Ef(x(T) - Ex(T)),$$

$$\text{subject to} \quad \begin{cases} \text{the wealth equation with } x(0) = x_0, \\ Ex(T) = z, \\ \pi(\cdot) \text{ admissible}, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ given

Examples of f

- $f(x) = x^2$: (continuous-time) Markowitz mean–variance model; studied extensively recently (Z. and Li 2000, Lim and Z. 2002, Lim 2004, Heunis and Labbe 2004, Bielecki, Jin, Pliska and Z. 2005...)
- $f(x) = \alpha x_+^2 + \beta x_-^2$: weighted mean–variance model
- $f(x) = x_-^2$: mean–semivariance model (studied extensively for single period)
- $f(x) = (x_-)^p$, $p > 0$: lower partial moment risk model (studied extensively for single period)
- $f(x) = |x|$: mean–absolute-deviation model (Konno and Yamazaki 1991 for single period)
- $f(x) = e^{-x}$: larger deviation more heavily penalized
- $f(x) = 0$ $\forall x \in \mathbb{R}_+$: downside risk model (studied extensively for single period)

A *déjà vu* of Utility Models?

NO!

- Most of the above f are not covered by the existing utility models where some standing assumptions are made
- In our model only the convexity or monotonicity of f is assumed
- The results, as will be demonstrated below, are fundamentally different

Solution: A Dual Approach

- Define

$$\rho(t) := \exp\left\{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)dW(s)\right\}$$

- Wealth equation is equivalent to

$$x(t) = \rho(t)^{-1}E(\rho(T)x(T)|\mathcal{F}_t), \quad \forall t \in [0, T]$$

- Budget constraint becomes $x_0 = E[\rho(T)x(T)]$
- The mean–risk problem is decomposed into two problems
- First problem: to find an optimal **terminal** wealth

$$\begin{array}{ll} \text{Minimize} & Ef(X - z), \\ \text{subject to} & \begin{cases} EX = z, \\ E[\rho(T)X] = x_0, \\ X \in L^2(\mathcal{F}_T, \mathbb{R}) \end{cases} \end{array} \tag{2}$$

- A **static** optimization problem!

A Dual Approach (Cont'd)

- Second problem: given the optimal solution X^* to (2), to find a portfolio that replicates X^*
- Equivalently, to find $(x(\cdot), \pi(\cdot))$ that solves the following equation

$$\begin{cases} dx(t) = [r(t)x(t)dt + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\ x(T) = X^*. \end{cases} \tag{3}$$

- An essentially backward stochastic differential equation (BSDE), solvable due to the assumption on $\sigma(t)$
- Hence the key is to solve the first problem

Weighted Mean–Variance Model

Let $f(x) = \alpha x_+^2 + \beta x_-^2$ where $\alpha > 0, \beta > 0$

Changing variable $Y := X - z$, the static problem (2) specializes to

$$\begin{array}{ll} \text{Minimize} & E(\alpha Y_+^2 + \beta Y_-^2), \\ \text{subject to} & \begin{cases} EY = 0, \\ E[\rho Y] = y_0, \\ Y \in L^2(\mathcal{F}_T, \mathbb{R}), \end{cases} \end{array} \tag{4}$$

where $\rho := \rho(T)$ and $y_0 := x_0 - zE\rho$

Introducing two Lagrange multipliers for the two constraints, one needs only to solve

$$\min_{Y \in L^2(\mathcal{F}_T, \mathbb{R})} E[\alpha Y_+^2 + \beta Y_-^2 - 2(\lambda - \mu\rho)Y] \tag{5}$$

Weighted MV Model (Cont'd)

Lemma 1. Problem (5) admits a unique optimal solution $Y^* = \frac{(\lambda - \mu\rho)_+}{\alpha} - \frac{(\lambda - \mu\rho)_-}{\beta}$.

Lemma 2. For any y_0, there exists a unique pair (λ, μ) such that the optimal solution Y^* in Lemma 1 satisfies $EY^* = 0, E[\rho Y^*] = y_0$. Moreover, $\lambda < 0, \mu < 0$ if $y_0 > 0$, $\lambda > 0, \mu > 0$ if $y_0 < 0$, and $\lambda = \mu = 0$ if $y_0 = 0$.

Solution to Weighted MV Problem

Theorem. The unique optimal portfolio for the weighted MV problem corresponding to (x_0, z) is given by

$$\pi^*(t) = (\sigma(t)')^{-1} y^*(t),$$

where $(x^*(\cdot), y^*(\cdot))$ is the unique solution to the BSDE

$$\begin{cases} dx(t) = [r(t)x(t) + \theta(t)y(t)]dt + y(t)'dW(t) \\ x(T) = \frac{(\lambda-\mu\rho)_+}{\alpha} - \frac{(\lambda-\mu\rho)_-}{\beta} + z, \end{cases}$$

with (λ, μ) being the unique solution to the system of algebraic equations

$$\begin{cases} \frac{E(\lambda-\mu\rho)_+}{\alpha} - \frac{E(\lambda-\mu\rho)_-}{\beta} = 0 \\ \frac{E[\rho(\lambda-\mu\rho)_+]}{\alpha} - \frac{E[\rho(\lambda-\mu\rho)_-]}{\beta} = x_0 - zE\rho. \end{cases}$$

Moreover, the minimum risk (weighted variance) value is given by

$$J^*(x_0, z) = -\mu(x_0 - zE\rho).$$

Some Remarks

Remark. If $z = \frac{x_0}{E\rho}$, then $\lambda = \mu = 0$ implying that $x^*(T) = z$ and $J^*(x_0, z) = 0$ under the optimal portfolio. Hence in this case the optimal portfolio is a risk-free portfolio. As a by-product, we have proved that a risk-free portfolio is available (which involves exposure to the stocks) even though the interest rate is random, and $E\rho$ is the discounting factor (indeed $E\rho = e^{-\int_0^T r(s)ds}$ when all the market parameters are deterministic).

Remark. When the market coefficients are deterministic, optimal portfolio can be obtained more explicitly via some Black–Scholes type equation.

Mean–Semivariance Model

Let $f(x) = x_-^2$

Define

$$\begin{aligned} \rho_0 &:= \inf\{\eta \in \mathbb{R} : P(\rho < \eta) > 0\}, \\ \rho_1 &:= \sup\{\eta \in \mathbb{R} : P(\rho > \eta) > 0\}. \end{aligned} \tag{6}$$

Remark. If $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(t)|^2 dt > 0$, then $\rho_0 = 0$ and $\rho_1 = +\infty$.

Theorem. The mean–semivariance problem does not admit an optimal solution so long as $z \neq \frac{x_0}{E\rho}$.

Idea of Proof

Main Idea: View the mean–semivariance problem as the limiting problem of the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha \to 0$

- It suffices to prove that the static optimization problem

$$\begin{array}{ll} \text{Minimize} & E(Y_-^2), \\ \text{subject to} & \begin{cases} EY = 0, \\ E[\rho Y] = y_0 \equiv x_0 - zE\rho, \\ Y \in L^2(\mathcal{F}_T, \mathbb{R}), \end{cases} \end{array} \tag{7}$$

  has no optimal solution

- Let $Y(\alpha)$ be the optimal solution to the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha > 0$

- If $y_0 < 0$, one can show that $E[Y(\alpha)_-^2] \to y_0^2/E(\rho - \rho_0)^2$ as $\alpha \downarrow 0$. However, for any feasible solution Y of (7), one can show via Cauchy–Schwartz's inequality that $E[Y_-]^2 > y_0^2/E(\rho - \rho_0)^2$. Hence there is no optimal solution
- If $y_0 > 0$, one can show that $E[Y(\alpha)_-^2] \to y_0^2/E(\rho_1 - \rho)^2$ as $\alpha \downarrow 0$, whereas $E[Y_-]^2 > y_0^2/E(\rho_1 - \rho)^2$ for any feasible solution Y. Again there is no optimal solution

Some Remarks

Remark. If $z = \frac{x_0}{E\rho}$, then there is a risk-free portfolio under which the terminal wealth is exactly z. This portfolio is therefore an optimal portfolio for the mean–semivariance problem.

Remark. Although the mean–semivariance problem in general does not admit optimal solutions, the infimum of the problem has been obtained explicitly. Specifically, the infimum is $\frac{y_0^2}{E(\rho-\rho_0)^2}$ if $y_0 < 0$, and is $\frac{y_0^2}{E(\rho_1-\rho)^2}$ if $y_0 > 0$. Moreover, asymptotically optimal portfolios can be obtained by replicating $Y(\alpha) + z$ as $\alpha \to 0$.

Mean–Downside-risk Model

Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}_-$, and $f(x) = 0 \ \forall x \in \mathbb{R}_+$ (an example: $f(x) = (x_-)^p$ for some $p > 0$ – *lower partial moment risk*)

Assumption. For any $0 \leq M_1 < M_2 \leq +\infty$,
$P\{\rho \in (M_1, M_2)\} > 0$ and
$P\{\rho = M_1\} = P\{\rho = M_2\} = 0.$

Remark. This assumption is satisfied when, say, $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(t)|^2 dt > 0$.

Theorem. The mean–down-risk model with f as the risk measure admits no optimal solution for any $z \neq \frac{x_0}{E\rho}$. On the other hand, if $z = \frac{x_0}{E\rho}$, then the model has an optimal portfolio which is the risk-free portfolio.

General Mean–Risk Model

Let f be convex, and strictly convex at 0

Define the subdifferential $\partial f(x)$ in the sense of convex analysis

$$\partial f(x) := \{x^* \in R : \ f(y) - f(x) \geq x^*(y-x), \ \ \forall y \in \mathbb{R}\}$$

We maintain the following assumption

Assumption. For any $0 \leq M_1 < M_2 \leq +\infty$,
$P\{\rho \in (M_1, M_2)\} > 0$ and
$P\{\rho = M_1\} = P\{\rho = M_2\} = 0.$

Solution

Theorem. One has the following conclusions regarding the solution to the mean–risk portfolio selection problem with the general f:

(i) Assume that either $\cup_{x\in\mathbb{R}}\partial f(x) = (-\infty, \bar{k}]$ or $\cup_{x\in\mathbb{R}}\partial f(x) = (-\infty, \bar{k})$ for some $\bar{k} \in \mathbb{R}$. If $\bar{\lambda} \notin \bar{\Lambda}$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in (\underline{y}, 0]$, where $\underline{y} = \lim_{\lambda\uparrow\bar{\lambda}} \tilde{g}(\lambda)$. If $\bar{\lambda} \in \bar{\Lambda}$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in \{\tilde{g}(\bar{\lambda})\} \cup (\underline{y}, 0]$. If in addition $\bar{\lambda} < \bar{k}$, then $\tilde{g}(\bar{\lambda}) = \underline{y}$.

(ii) Assume that either $\cup_{x\in\mathbb{R}}\partial f(x) = [\underline{k}, \infty)$ or $\cup_{x\in\mathbb{R}}\partial f(x) = (\underline{k}, \infty)$ for some $\underline{k} \in \mathbb{R}$. If $\underline{\lambda} \notin \underline{\Lambda}$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in [0, \bar{y})$, where $\bar{y} = \lim_{\lambda\downarrow\underline{\lambda}} \tilde{g}(\lambda)$. If $\underline{\lambda} \in \underline{\Lambda}$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in \{\tilde{g}(\lambda)\} \cup [0, \bar{y})$. If in addition $\underline{\lambda} > \underline{k}$, then $\tilde{g}(\bar{\lambda}) = \bar{y}$.

(iii) Assume that $\cup_{x\in\mathbb{R}}\partial f(x) = \mathbb{R}$. Then the problem admits an optimal solution if and only if $x_0 - zE\rho \in A \cup B$, where

$$A = \begin{cases} [\underline{y}, 0], & \text{if } \bar{\lambda} \in \Lambda \\ (\underline{y}, 0], & \text{if } \bar{\lambda} \notin \Lambda \end{cases}, \qquad B = \begin{cases} [0, \overline{y}], & \text{if } \underline{\lambda} \in \Lambda \\ [0, \overline{y}), & \text{if } \underline{\lambda} \notin \Lambda. \end{cases}$$

(iv) Assume that there exists $M_1, M_2 \in \mathbb{R}$ such that $\cup_{x\in\mathbb{R}}\partial f(x) \subset [M_1, M_2]$. Then the problem admits an optimal solution if and only if $z = x_0/E\rho$.

Examples

Example. $f(x) = |x|$ (mean–absolute-deviation model). f is strictly convex at 0, and $\cup_{x\in\mathbb{R}}\partial f(x) = [-1, 1]$. Thus the continuous-time mean–absolute-deviation model admits an optimal solution if and only if $z = x_0/E\rho$, in which case the optimal portfolio is simply the risk-free one.

Example. $f(x) = e^{-x}$ (larger deviation of the terminal wealth from its mean is more heavily penalized). f is strictly convex, $\cup_{x\in\mathbb{R}}\partial f(x) = (-\infty, 0)$. The mean–risk portfolio selection problem admits an optimal solution if and only if $x_0 - zE\rho \in [(E\rho)(E\ln\rho) - E(\rho\ln\rho), 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0-(E\rho)(E\ln\rho)+E(\rho\ln\rho)}{E\rho}]$. When the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim $z - \ln(-\lambda + \mu\rho)$ where (λ, μ) is the unique solution pair to the following algebraic equation (which must admit a solution):

$$\begin{cases} E\ln(-\lambda + \mu\rho) = 0, \\ E[\rho\ln(-\lambda + \mu\rho)] = zE\rho - x_0. \end{cases}$$

Examples (Cont'd)

Example. $f(x) = (x-1)_-^2$ (shift of the mean–semivariance model). f is not strictly convex everywhere; but it is indeed strictly convex at 0. $\cup_{x\in\mathbb{R}}\partial f(x) = (-\infty, 0]$. The original portfolio selection problem admits an optimal solution if and only if $x_0 - zE\rho \in [E\rho - E\rho^2/E\rho, 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0}{E\rho} + \frac{E\rho^2}{(E\rho)^2} - 1]$. When the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim $z + 1 + \frac{\lambda - \mu\rho}{2}$ where (λ, μ) is the unique solution pair to the following *linear* algebraic equation:

$$\begin{cases} \lambda - \mu E\rho = -2, \\ \lambda E\rho - \mu E\rho^2 = 2x_0 - 2(1+z)E\rho. \end{cases}$$

Asymptotic Optimal Portfolios

In all the cases when optimal portfolios do **not** exist, consider a perturbed risk function

$$f_\alpha(x) = f(x) + \alpha|x|^2, \quad \alpha > 0$$

It can be shown:

- The mean–risk portfolio selection problem with f_α **must** admit an optimal solution
- The corresponding optimal portfolios π_α is asymptotically optimal for the original problem when $\alpha \downarrow 0$

Fixed-Target Problem

$$\text{Minimize} \quad Ef(x(T) - Z),$$

$$\text{subject to} \quad \begin{cases} \text{the wealth equation with } x(0) = x_0, \\ \pi(\cdot) \text{ admissible,} \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$, and $Z \in L^2(\mathcal{F}_T, \mathbb{R})$

Back to Single Period

- Single period
- $m(\geq 2)$ securities, each with total return R_j, $j = 1, \cdots, m$
- $r_j = E[R_j]$, $E[R_j^2] < \infty$
- An agent with fund x_0, and a targeted expected payoff z at the end of the period
- To find a portfolio $\pi = (\pi_1, \cdots, \pi_m)$ so as to

$$\begin{array}{ll} \text{Minimize} & E\left[\left(\sum_{i=1}^m \pi_i R_i - \sum_{i=1}^m \pi_i r_i\right)_-\right]^2 \\ \text{subject to} & \sum_{i=1}^m \pi_i = x_0 \\ & \sum_{i=1}^m r_i \pi_i = z \end{array} \tag{8}$$

Existence of Efficient Portfolios

- Vast literature on single-period mean–semivariance models
- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean–variance model
- Existence of efficient portfolios/frontier not addressed
- Technically non-trivial as the feasible region generally **unbounded**
- Direct motivation: continuous-time case!

What about Mean–Variance?

- If $\sigma := (\sigma_{ij})$ non-singular, then the objective function is coercive; and hence there is optimal solution
  - $f : \mathbb{R}^d \to \mathbb{R}$ called *coercive* if $\lim_{|x| \to +\infty} f(x) = +\infty$
- If σ singular, the existence is ensured by the Frank–Wolfe theorem
  - Frank–Wolfe theorem (1956): any *quadratic* function bounded below on a nonempty polyhedron must achieve its infimum on the polyhedron

A Lemma

Consider

$$\min_{x \in \mathbb{R}^d} E[(A + B'x)_-]^2, \tag{9}$$

where $B \equiv (B_1, \cdots, B_d)'$, and A, B_i are random variables with $EA^2 < +\infty, EB_i^2 < +\infty$, $i = 1, \cdots, d$.

Lemma 3. If $EB_i = 0$, $i = 1, \cdots, d$, then problem (9) admits optimal solutions.

Remark. Assumption that each B_i has zero mean is crucial. Let $A = -1, B = (e^{W_1}, \cdots, e^{W_d})$, where $(W_1, \cdots, W_d)$ follows $N(0, I_d)$. For any $0 \neq x \in \mathbb{R}_+^d$, $\lim_{\alpha \to +\infty} E[(A + B'(\alpha x))_-]^2 = 0$. This implies the optimal value of (9) is zero. However, this value cannot be achieved since $E[(A + x'B)_-]^2 > 0$ for any $x \in \mathbb{R}^d$.

Sketch of Proof

Without loss of generality, assume that $B_1, \cdots, B_d$ are linearly independent, namely, $\alpha_1 = \cdots = \alpha_d = 0$ whenever $\sum_{i=1}^m \alpha_i B_i = 0$ for real numbers $\alpha_1, \cdots, \alpha_d$.

- Define $S := \{(k, y) \in \mathbb{R}^{d+1} : 0 \leq k \leq 1, |y| = 1\}$, $l := \inf_{(k,y)\in S} E[(kA + B'y)_-]^2$. There exists $(k^*, y^*) \in S$ such that $l = E[(k^*A + B'y^*)_-]^2$.
- If $l = 0$, then $k^* > 0$ (if $k^* = 0$, then $E[(B'y^*)_-]^2 = 0$ which yields $B'y^* \geq 0$. However, $E[B'y^*] = E[B']y^* = 0$. So $B'y^* = 0$, a contradiction), and $x^* := y^*/k^*$ is an optimal solution.
- If $l > 0$, then for any $x \in \mathbb{R}^d$ with $|x| \geq 1$, we have

$$E[(A + B'x)_-]^2 = |x|^2 E[(\frac{A}{|x|} + \frac{x'}{|x|}B)_-]^2 \geq l|x|^2.$$

Existence for Single-Period M–S

Theorem. For any $x_0 \in \mathbb{R}$ and $z \in \mathbb{R}$, problem (8) admits optimal solutions if and only if it admits feasible solutions.

Idea of Proof. Let $\xi_i = R_i - r_i$. After eliminating x_1 and x_2 from the constraints, one gets the following equivalent problem

$$\min_{(x_3,\cdots,x_m)\in\mathbb{R}^{m-2}} E\left[(A + \sum_{i=3}^{m} x_i B_i)_-\right]^2 .$$

where

$$A = a\xi_1 + \tfrac{z-ar_1}{r_2-r_1}(\xi_2 - \xi_1),$$
$$B_i = \xi_i - \xi_1 - (r_i - r_1)\tfrac{\xi_2-\xi_1}{r_2-r_1}.$$

Then Lemma 3 applies.

Future Research

- Incomplete market (the second problem becomes significant)
- Other risk measures: safety first, VaR, minimax,...

Credits

This talk based on the following two papers:

- Hanqing Jin, Jia-An Yan, and Xun Yu Zhou, *Ann. l'Inst. Henri Poincaré*, En Hommage à Paul-André Meyer (1934–2003), 2005
- Hanqing Jin, Harry Markowitz, and Xun Yu Zhou, *Math. Finance*, Special Issue of Yellow Mountain Workshop, 2006