Valuation of Credit Derivatives

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Credit Risk Valuation Problems

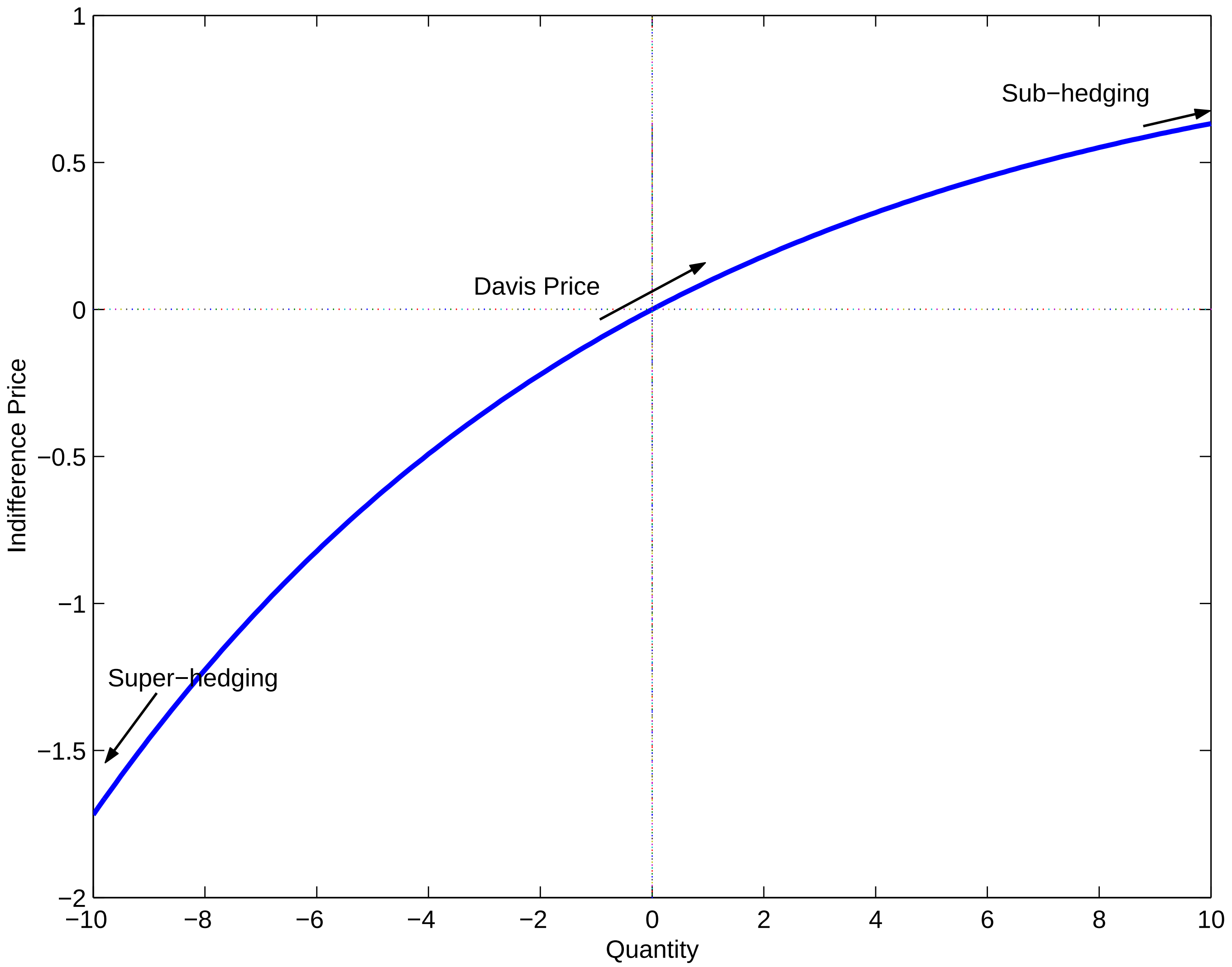
- Intensity-based models of default risk are highly popular.
- They capture surprise elements of credit events.
- They reproduce significant *yield spreads* at short maturities.
- Low-dimensional models are (relatively) easy to calibrate from market defaultable bond/CDS data.
- Pricing of exotic credit derivatives builds on technology of interest rate models.

Difficulties

- Scaling up to multiple names – correlation of default times.
- Exotic products depend on # of defaults (or loss distribution) of a basket of corporate bonds.
- Market prices as though there is strong clustering of defaults.
- Dimension as high as 300 (CDOs), then CDO^2 , CDO^3.
- Copulas used to correlate default times: non-dynamic, low dimensional in practice.

Utility Indifference Derivative Pricing

- Dynamic generalization of certainty equivalent .
- Reasonable preference-based valuation methodology in illiquid/OTC markets.
- *E.g.* options on non-traded assets, weather derivatives; (PUP book on indifference pricing , 2005).
- Computationally tractable (and wealth-independent) under exponential utility .
- Nonlinear pricing rule.
- Credit & Indifference Pricing : see also Collin-Dufresne *et al.*, Bielecki-Jeanblanc-Rutkowski, Becherer & Schweizer.


Sub−hedging
Davis Price
Super−hedging
Indifference Price
Quantity
1
0.5
0
−0.5
−1
−1.5
−2
−10
−8
−6
−4
−2
0
2
4
6
8
10

Typical Single-Name Intensity Models

- All models under pricing measure $\mathbb{P}^{\star}$.

- Default time τ is first jump of a time-changed (standard) Poisson process:

$$N\left(\int_0^t \lambda_s \, ds\right),$$

where N and λ are independent.

- Draw $\xi \sim \mathrm{EXP}(1)$, then

$$\tau = \inf\left\{t : \int_0^t \lambda_s \, ds = \xi\right\}.$$

- *E.g.*: λ is a diffusion (CIR).

Defaultable Bond Pricing

- Payoff $\mathbf{1}_{\{\tau > T\}}$.
- Price

$$
\begin{aligned}
P_0(T) &= \mathbb{E}^\star\left\{e^{-rT}\mathbf{1}_{\{\tau>T\}}\right\} \\
&= e^{-rT}\mathbb{P}^\star\{\tau > T\} \\
&= \mathbb{E}^\star\left\{\exp\left(-\int_0^T (r+\lambda_s)\,ds\right)\right\}.
\end{aligned}
$$

- Same structure as *short rate models*.
- Yield spread: $P_0(T) = \exp(-(r + Y(T))T)$:

$$
Y(T) = -\frac{1}{T}\log\left(\frac{P_0(T)}{e^{-rT}}\right).
$$

Issues

- Intensity models resolve a major shortcoming of (constant volatility) structural models: yield spreads not small at short maturities.
- *E.g.*: for λ constant, $Y(T) = \lambda$.
- Loss of economic intuition – why a default? No direct relation to firm's stock price.
- While single name default time models can be calibrated, how to deal with *joint distributions*?
- How to compute with ~ 300 names?

Capturing essential effects of stochastic intensities

- Stochastic intensity like (independent) stochastic volatility for Poisson process.
- Distributional models: CIR

$$d\lambda_t = \alpha(m - \lambda_t)\,dt + \beta\sqrt{\lambda_t}\,dW_t^\star.$$

- Scales : $\lambda_t = \lambda(Y_t)$
  - SLOW $dY_t = \delta b(Y_t)\,dt + \sqrt{\delta}a(Y_t)\,dW_t^\star.$
  - FAST $dY_t = \frac{1}{\varepsilon}(m - Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\,dW_t^\star.$
  - No arbitrage price

$$\approx \text{const. } \lambda \text{ price} + \begin{pmatrix} \sqrt{\varepsilon} \\ \sqrt{\delta} \end{pmatrix} \text{stochastic intensity correction}.$$

Indifference Pricing

- Given a utility function U, at what value is the buyer indifferent in terms of maximum expected utility between having and not having the derivative?
- Take $U(x) = -e^{-\gamma x}$, with $x \in \mathbb{R}$.
- Stock price S and intensity λ:

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW^{(1)} \\ \lambda &= \lambda(Y) \\ dY &= b(Y)\,dt + a(Y)\left(\rho\,dW^{(1)} + \sqrt{1-\rho^2}\,dW^{(2)}\right). \end{aligned}$$

- Wealth process X:

$$\begin{aligned} dX &= \pi\frac{dS}{S} + r(X-\pi)\,dt \\ &= (rX + \pi(\mu - r))\,dt + \sigma\pi\,dW^{(1)}. \end{aligned}$$

- Want

$$\sup_{\pi} I\!\!E \left\{ -e^{-\gamma(e^{-rT}X_T)} \mathbf{1}_{\{\tau > T\}} + (-e^{-\gamma(e^{-r\tau}X_\tau)}) \mathbf{1}_{\{\tau \le T\}} \right\}.$$

- Switch to discounted variable $X_t \mapsto e^{-rt}X_t$ and $\mu \mapsto \mu - r$. Value function

$$M(t,x,y) = \sup_{\pi} I\!\!E_{t,x,y} \left\{ -e^{-\gamma X_T} \mathbf{1}_{\{\tau > T\}} + (-e^{-\gamma X_\tau}) \mathbf{1}_{\{\tau \le T\}} \right\}$$

solves

$$M_t + \mathcal{L}_y M - \frac{(\mu M_x + \rho\sigma a M_{xy})^2}{2\sigma^2 M_{xx}} + \lambda(y)(-e^{-\gamma x} - M) = 0,$$

with $M(T,x,y) = -e^{-\gamma x}$.

- Reduce to

$$M(t,x,y) = -e^{-\gamma x} u(t,y)^{1/(1-\rho^2)},$$

where

$$u_t + \widetilde{\mathcal{L}}_y u - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) u + (1-\rho^2)\lambda(y) u^{-\theta} = 0,$$

with $u(T,y) = 1$ and

$$\theta = \frac{\rho^2}{1-\rho^2}.$$

- Reaction-diffusion equation.

Add claim $\mathbf{1}_{\{\tau>T\}}$

- Define $c = e^{-rT}$. Value function

$$H(t,x,y) = \sup_{\pi} \mathbb{E}_{t,x,y}\left\{-e^{-\gamma(X_T+c)}\mathbf{1}_{\{\tau>T\}} + (-e^{-\gamma X_\tau})\mathbf{1}_{\{\tau\leq T\}}\right\}.$$

- Reduce $H(t,x,y) = -e^{-\gamma(x+c)}w(t,y)^{1/(1-\rho^2)}$, to

$$w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) w + (1-\rho^2)e^{\gamma c}\lambda(y)w^{-\theta} = 0,$$

with $w(T,y) = 1$. A similar reaction-diffusion equation.

- Indifference price: $M(0,x,y) = H(0,x-p,y)$ given by

$$p = e^{-rT} - \frac{1}{\gamma(1-\rho^2)}\log(w/u).$$

By comparison principles: $u < w$ as $e^{\gamma c} > 1$.

Constant Intensity Case

- When intensity λ is fast or slow , we expand, and the principal term is the *constant intensity* case, either $\langle\lambda\rangle$ or $\lambda(y)$.

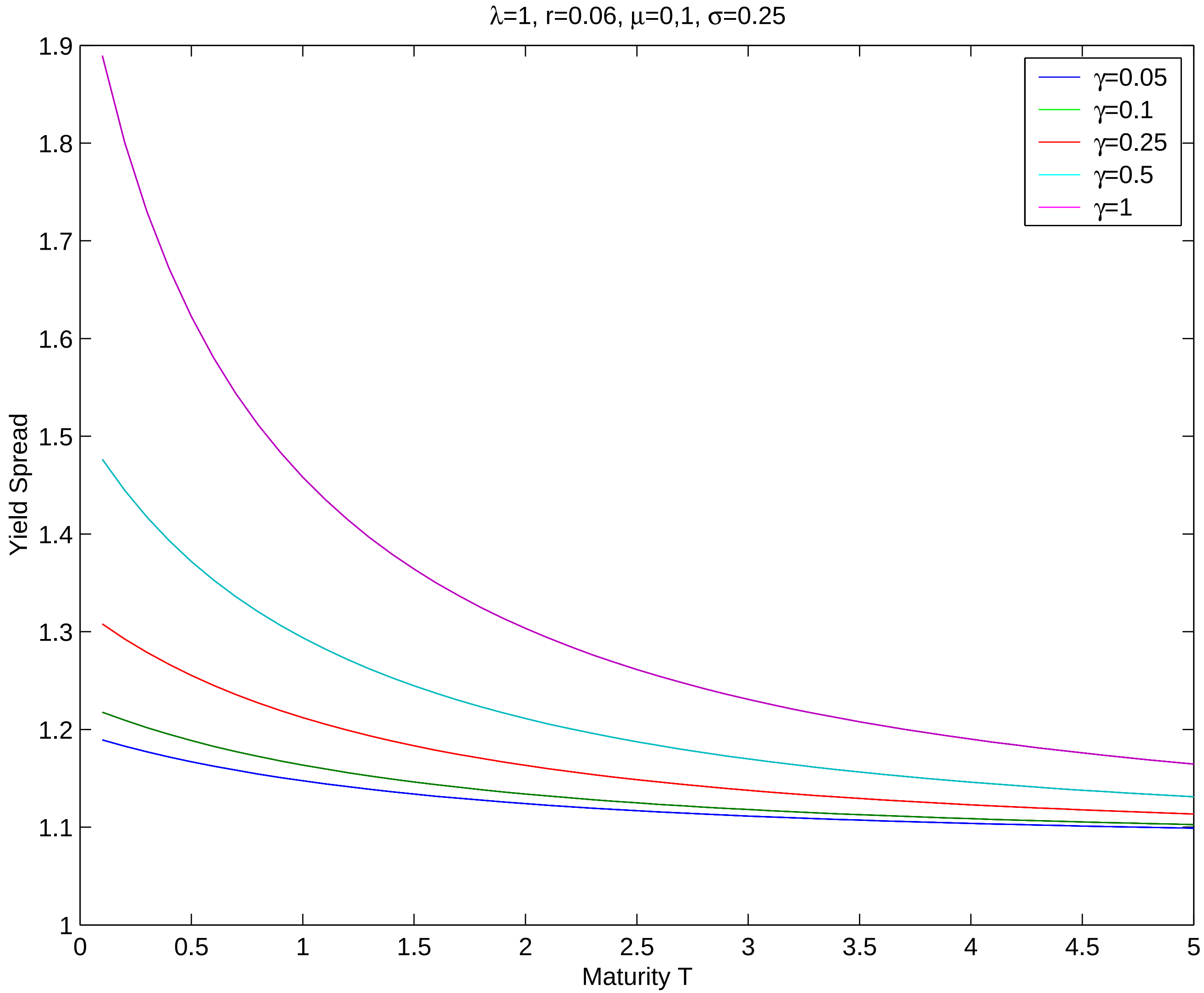
- When λ is constant, defaultable bond price is

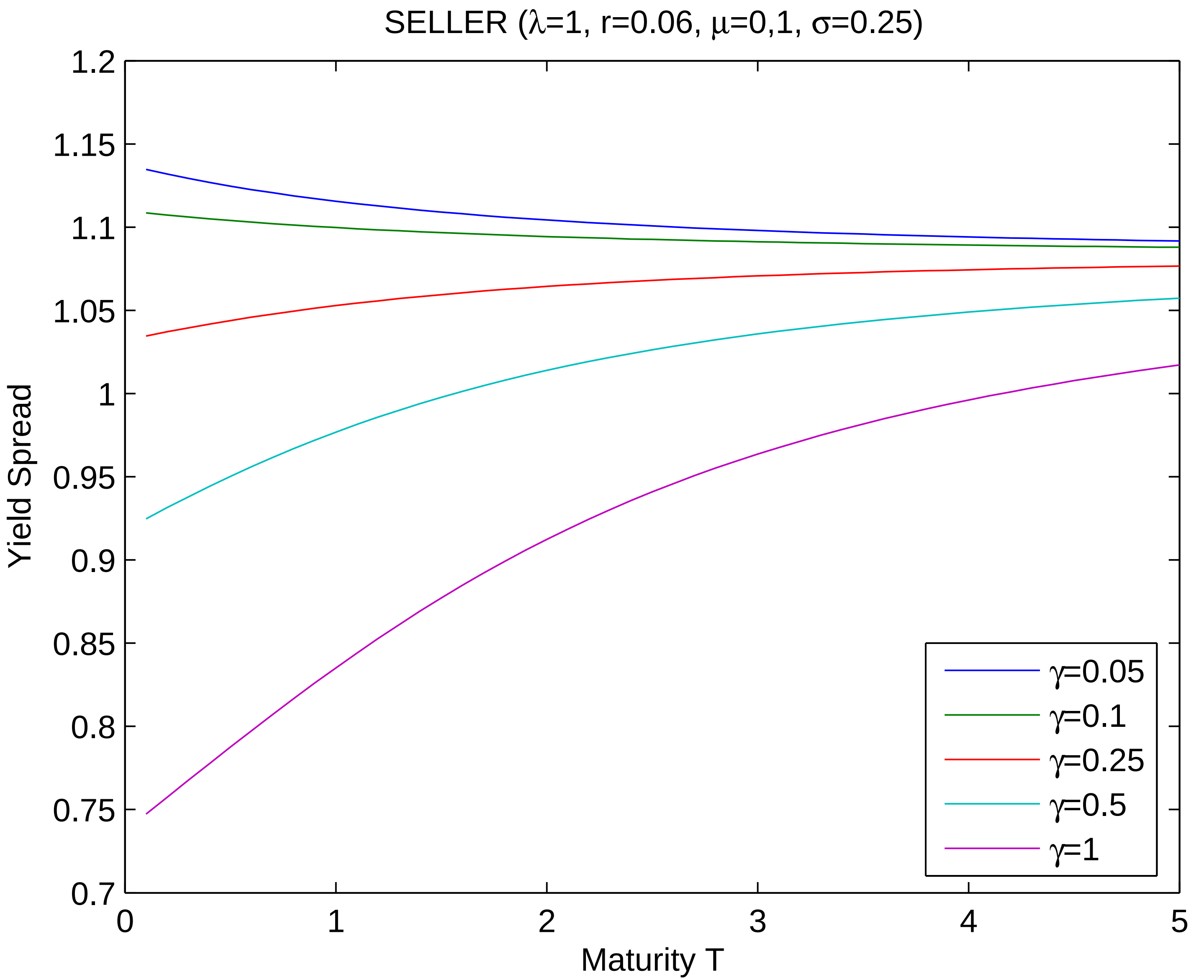
$$p_0(T) = \frac{1}{\gamma}\log\left(\frac{1+\beta}{1+\beta e^{-\gamma c}}\right),$$

with

$$\beta = \frac{\lambda}{\alpha}\left(\frac{e^{-\alpha T}}{1-e^{-\alpha T}}\right), \qquad \alpha = \frac{\mu^2}{2\sigma^2} + \lambda.$$

- Plot of yield spread $Y_0(T) = -\frac{1}{T}\log(p_0(T)/e^{-rT})$.


λ=1, r=0.06, μ=0,1, σ=0.25
Yield Spread
Maturity T
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
γ=0.05
γ=0.1
γ=0.25
γ=0.5
γ=1


SELLER (λ=1, r=0.06, μ=0,1, σ=0.25)
Yield Spread
Maturity T
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
0.75
0.7
0
1
2
3
4
5
γ=0.05
γ=0.1
γ=0.25
γ=0.5
γ=1

Multi-Name Case

- N firms. Stock prices processes

$$\frac{dS^{(i)}}{S^{(i)}} = \mu_i \, dt + \sum_{j=1}^{N} \sigma_{ij} \, dW^{(j)}.$$

Discounted wealth process

$$dX = \sum_i \pi_i \mu_i \, dt + \sum_{i,j} \pi_i \sigma_{ij} \, dW^{(j)}.$$

- Merton value function $M^{(N)}(t,x)$ solves

$$M_t^{(N)} - \frac{1}{4}(\mu^T A^{-1} \mu) \frac{(M_x^{(N)})^2}{M_{xx}^{(N)}} + \sum_i \lambda_i \left(M_i^{(N-1)} - M^{(N)} \right) = 0,$$

where $A = \sigma \sigma^T$, and $M_i^{(N-1)}$ is the Merton value function when firm i has dropped out.

- After $M^{(N)}(t,x) = -e^{-\gamma x} v^{(N)}(t)$, have ODE

$$\frac{d}{dt}v^{(N)} - \left(\frac{1}{4}(\mu^T A^{-1}\mu) + \sum \lambda_i\right) v^{(N)} + \sum \lambda_i v_i^{(N-1)} = 0,$$

with $v^{(N)}(T) = 1$. Similar for case with claim.

- However, still have combinatorial problem of many value functions. Unless we assume *exchangeability*, when k firms have defaulted, have to solve the Merton problems for each of

$$\left(\begin{array}{c} N \\ k \end{array} \right)$$

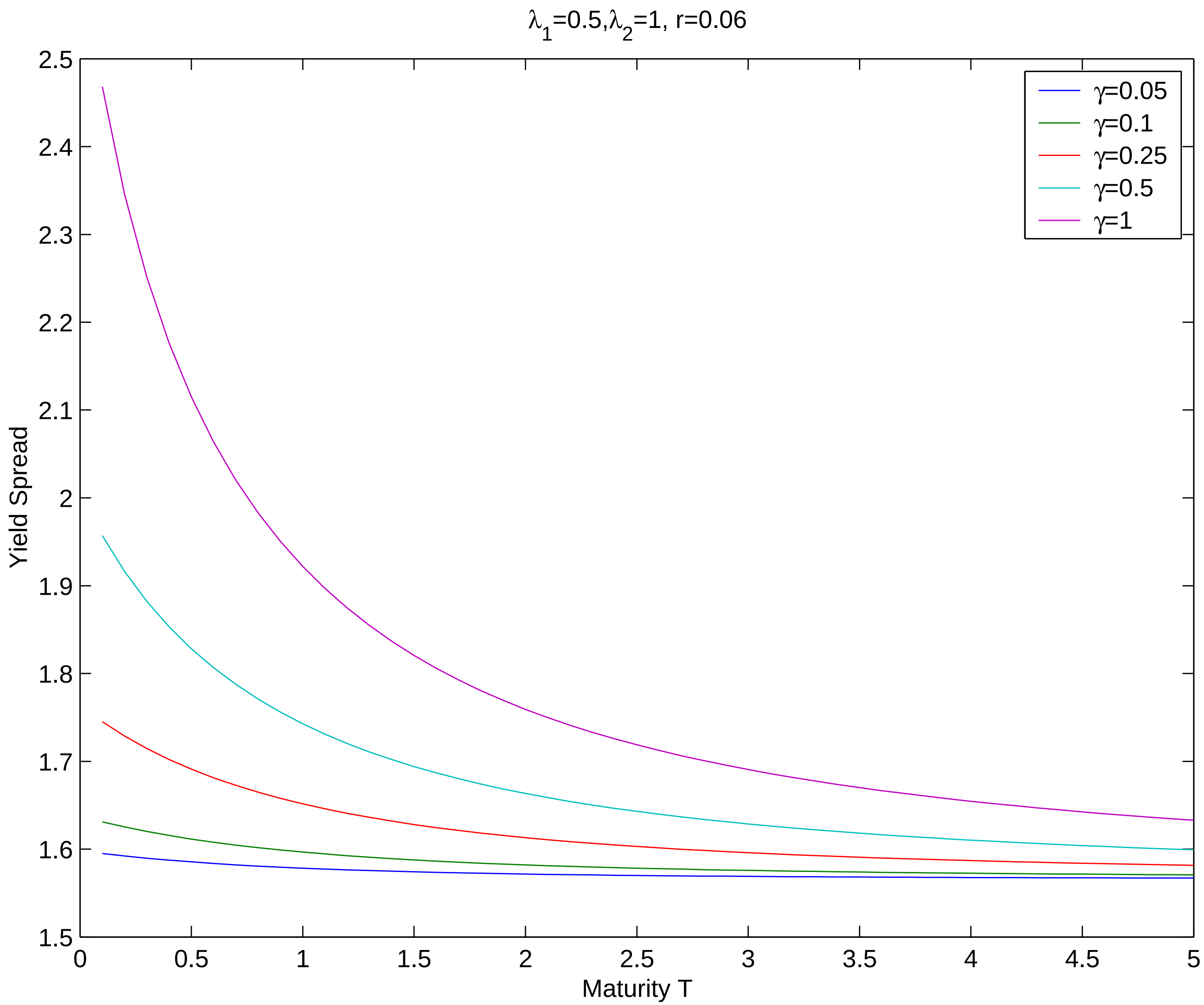
combinations of possible firms left.

Two-Name Case

- $N = 2$, Independent defaults at rates λ_1 and λ_2 .
- If $\tau = \min(\tau_1, \tau_2)$, no arbitrage valuation is

$$e^{-rT} I\!P^{\star}\{\tau > T\} = e^{-(r+\lambda_1+\lambda_2)T},$$

  so yield spread is $\lambda_1 + \lambda_2$.
- Solve value functions when either firm i is left, with and without the claim, then solve for both firms left, again with and without the claim.
- Plot the yield spread.


λ1=0.5,λ2=1, r=0.06
γ=0.05
γ=0.1
γ=0.25
γ=0.5
γ=1
Yield Spread
Maturity T
2.5
2.4
2.3
2.2
2.1
2
1.9
1.8
1.7
1.6
1.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Single Name Asymptotics

- Intensity $\lambda(Y_t)$ where Y is *fast mean-reverting* OU:

$$dY_t = \frac{1}{\varepsilon}(m - Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho\,dW^{(1)} + \sqrt{1-\rho^2}\,dW^{(2)}\right),$$

while $dS = \mu S\,dt + \sigma S\,dW^{(1)}$.

- Merton value function: $M(t,x,y) = -e^{-\gamma x}u(t,y)^{1/(1-\rho^2)}$:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)u + (1-\rho^2)\lambda(y)u^{-\theta} = 0;$$

and $u(T,y) = 1$, where

$$\begin{aligned}
\mathcal{L}_0 &= \nu^2\partial_y^2 + (m-y)\partial_y \\
\mathcal{L}_1 &= -\frac{\rho\mu\nu\sqrt{2}}{\sigma}\partial_y \\
\mathcal{L}_2 &= \partial_t - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right).
\end{aligned}$$

Approximation

- Construct expansion

$$u(t,y) = u_0(t) + \sqrt{\varepsilon} u_1(t) + \varepsilon u_2(t,y) + \cdots .$$

Find u_0 explicitly as before, with constant "averaged" intensity $\bar{\lambda} = \langle \lambda \rangle$.

- u_1 solution of ODE

$$u_1' - \left((1-\rho^2)\bar{\alpha} + \rho^2 \bar{\lambda} u_0^{-\delta}\right) u_1 = k(u_0^{-\theta} - u_0)$$

with $u_1(T) = 0$.

- Numerical (or hypergeom. function).

Approximation of buyer's value function

- Similarly for buyer's value function

$$H(t,x,y) = -e^{-\gamma(x+c)} w(t,y)^{1/(1-\rho^2)}$$

  we expand $w(t,y) = w_0(t) + \sqrt{\varepsilon} w_1(t) + \varepsilon u_2(t,y) + \cdots$. Find w_0 explicitly as before.

- w_1 solution of ODE

$$w_1' - \left((1-\rho^2)\bar{\alpha} + \rho^2 e^{\gamma c} \bar{\lambda} w_0^{-\delta}\right) w_1 = k(e^{\gamma c} w_0^{-\theta} - w_0)$$

  with $w_1(T) = 0$.

- Then indifference price

$$p \approx e^{-rT} - \frac{1}{\gamma(1-\rho^2)} \log\left(\frac{w_0 + \sqrt{\varepsilon} w_1}{u_0 + \sqrt{\varepsilon} u_1}\right).$$

Concluding Remarks

- Non-trivial yield spreads even from constant intensities.
- Nonlinearity of indifference pricing rule acts as a correlator of default times via the effect of risk-aversion on portfolios.
- Computational/combinatorial problem remains, but under constant intensities deal with ODEs.
- Can stretch to asymptotic correction for stochastic intensity in fast and slow fluctuation regimes to alter short and long ends of spread curves.
- In multi-name case, "correlation effect" of indifference pricing corrected by correlation between diffusion intensities.