A Tale of Two Growths: Modeling Stochastic Endogenous Growth and Growth Stocks

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(Joint Work with Samuel Kou at Harvard University)

1 Introduction

Related Papers.

Mathematical Background:

1. Advances in Applied Probability, 2003

2. Math. of Operations Research, 2004

This paper is about the economic model

Growth Stocks

- The components of growth stocks may vary over time (perhaps consisting of railroad and utility stocks in the early 1900's, and biotechnology and internet stocks in 2000's).

- Studying the general properties of growth stocks is important to understand financial markets and economic growth over time, as growth stocks often represent the most innovative industries.

Difficulties: Growth stocks tend to have

- low or even negative earnings,
- high volatility,
- unpredictable future earnings,
- and many intangible assets (such as patents and human capital)

It is a great challenge to derive a meaningful mathematical model within the traditional valuation framework in finance.

Two puzzles

1. The WSJ article.

2. Growth Rate

Analyst Discovers Order in the Chaos Of Huge Valuations for Internet Stocks

HEARD ON THE STREET

By Greg Ip

Staff Reporter of The Wall Street Journal

Internet stocks, the conventional wisdom goes, are a chaotic mishmash defying any rules of valuation.

But one unconventional analyst thinks he has found proof of precisely the opposite: that Internet stocks adhere to a mathematical valuation system so rigid, it resembles patterns found in nature. That pattern suggests there may be fewer ultimate winners in the Internet arena than some investors expect.

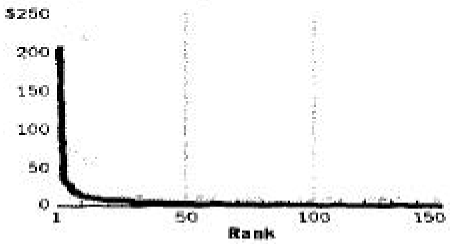
The analyst, Michael Mauboussin, the chief investment strategist at Credit Suisse First Boston, found this pattern, not by comparing Internet stocks to more ordinary benchmarks such as earnings, but by looking at the valuation of Internet stocks relative to one another. Among Internet stocks, he says, "there is literally a mathematical relationship between the ranking of the stock and its capitalization."

Most conventional analysis finds the sector's enormous valuation irrational. Mr. Mauboussin doesn't try to justify the high prices investors are willing to pay for Internet stocks as a group. Instead, he is intrigued by how so much of the stock-market value of Internet concerns is clustered into just a handful of companies. Indeed, just 1% of 400 companies in the sector account for 40% of its $900 billion value in the stock market.

A Method to the Internet Madness?

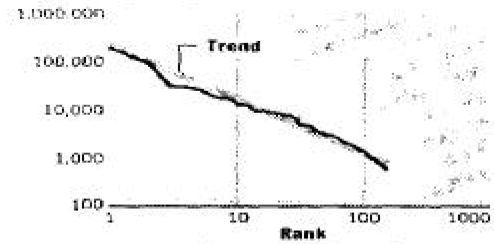
Internet Stock Values Appear Jumbled...

Internet stock values in billions, and rank on normal scale



But May Adhere to a Hidden Order

Internet stock values and rank on logarithmic scale



Source: Credit Suisse First Boston

Mr. Mauboussin uses some mathematical wizardry to find the pattern. Simply scattering the companies' value and ranking (relative to one another) on an ordinary chart results in a hockey-stick-like pattern: a couple of companies in the multiple billions of dollars, and everything else clustered close to zero.

Using some slightly different but relatively simple math, he found that each company's value bears a predictable relationship to the others'. He couldn't find a pattern like this in any other sector.

"I don't know why it works for this sector, but it suggests a couple of things," Mr. Mauboussin says. "One, there is a little method to the madness of how the market values these things. More important, these are winner-take-all or winner-take-most markets."

The pattern emerges when the companies' values are plotted along with their market-capitalization rank on a logarithmic chart. On such a chart, each inch equals a similar percentage change. For example, the distance between 100 and 1,000 is the same as that between 1,000 to 10,000, because both moves represent a 900% increase. Mr. Mauboussin says this is characteristic of a "power law," normally found only in measuring natural or social

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Growth Rate Puzzle

	Period	Annual Growth
Japan Telecom	68-04	4.01%
UK Telecom	84-04	1.84%
US Telecom	71-04	1.42%
NASDAQ	65-04	3.53%

Two objectives of the current paper:

- to construct a new stochastic endogenous growth model.

- to show that the stochastic endogenous growth model may provide a useful framework to understand several issues related to growth stocks.

Endogenous growth theory aims at replacing the assumption of exogenous technological progress in Solow's (1956) neoclassical growth theory by endogenous technological progress.

A. Stochastic Endogenous Growth

Various deterministic models have been proposed for endogenous technological progress:

- the AK style growth models in Romer (1986) and Lucas (1988)

- the quality improvement models in Grossman and Helpman (1991a, 1991b, 1991c)

- the creative destruction model in Aghion and Howitt (1992)

- and the endogenous R&D growth model with monopolistic competition first proposed in Romer (1990) (modified later in Jones, 1995a).

For surveys, see the books by Aghion and Howitt (1998), Barro and Sala-i-Martin (1995), Jones (2002a), Lucas (2002), and Ljungqvist and Sargent (2000).

While most of the endogenous growth models are deterministic, several stochastic endogenous growth models, though, have been proposed in the literature; see for example Stradler (1990) and Stiglitz (1994).

Compared with the few existing stochastic endogenous growth models, the model that we shall construct provides an interesting link between economic growth and the valuation of growth stocks.

B. Growth Stocks

We shall see in this paper that the study of stochastic endogenous growth may lead to an understanding of several issues related to growth stocks, namely

- how to derive a meaningful economic model for growth stocks;
- how to understand the "boom and burst" phenomenon of growth stocks
- what should be the appropriate long-run average returns of growth stocks.

C. Outline of the Model

We first extend the deterministic endogenous technological growth model in Romer (1990) and Jones (1995a) to a stochastic one, which is then used to build a model for growth stocks.

We extend the deterministic Romer model because it provides a natural link between growth stocks, monopolistic competition, and labor growth.

D. The contribution of the current paper

(a) Introduce randomness in Romer's model.

For example, to get an equilibrium labor allocation among research and production sectors, one can only match the wage for R&D labor in the production sector with the *average* wage (but not the *individual* wages) in the research sector, because of randomness.

This contrasts with the deterministic Romer model in which the wages within the research sector are all the same.

Furthermore, the random shock to the whole economy affects all main economic variables, such as wages, productions, stock prices, etc.

(b) Provide several interesting insights for growth stocks.

(1) The prices of growth stocks can be highly volatile, and have a boom-and-burst behavior.

(2) The long-run average return of growth stocks is surprisingly small — in fact, it is just the growth rate of R&D labor in the economy.

(3) Cross-sectionally, if one plots the logarithm of the market capitalization of large cap growth stocks against their associate ranks, one should expect to see an almost linear pattern.

This explains an empirically observed size distribution puzzle reported in the Wall Street Journal, December 29, 1999, for internet stocks:

$$\log M_{(j)}(t) \sim \log j$$

2 Implication of the Model on Growth Stocks

2.1 A Decomposition of Growth Stocks

The total equity value of a growth firm $M_j(t)$ (which becomes the market capitalization of the growth firm if the firm is publicly traded and has no debt) is given by

$$M_j(t) = const \cdot \ \theta(t)[H(t)]^{\frac{\alpha}{\alpha+\beta}}[L(t)]^{\frac{\beta}{\alpha+\beta}}\left(\xi_j(t)\right)^{\frac{1-\alpha-\beta}{\alpha+\beta}}$$

Besides the constant term, the value of a growth stock is affected by the economy-wise shock $\theta(t)$, the individual research effectiveness $\xi_j(t)$, the R&D labor size $H(t)$, and the non R&D labor size $L(t)$.

2.2 Dynamic Implications

(1) Because both $\theta(t)$ and $\xi_j(t)$ are mean reverting processes, the growth rate of the expectation of $M_j(t)$, $E(M_j(t))$, is determined by the growth rates of $H(t)$ and $L(t)$.

For example, in U.S. the share of R&D labor from about 0.25% in 1950's to about 1% in 1990's, but at the same time the total labor force (hence the non R&D labor force) only changed mildly.

Therefore, the model suggests that the growth rate for growth stocks should be roughly equal to that of the R&D labor.

(2) Since the research effectiveness is very unpredictable, the volatility of $\xi_j(t)$ is in general quite high, which also leads to high volatility of $M_j(t)$. In other words, the price of growth stocks can display a "boom-and-burst" behavior solely due to the high volatility.

(3) The value of a growth stock fluctuates in harmony with the overall economic condition $\theta(t)$. In other words, in the economic downturn, i.e. when $\theta(t)$ is small, the price of a growth stock price should also be low.

Both the second and the third dynamic implications seem to make intuitive sense. The first implication, however, is more interesting.

To study it closely, we will present a numerical illustration (though not an empirical test).

As a proxy for the equity values of growth stocks, we take the data of telecommunication indices in Japan, U.K., and U.S., as well as the NASDAQ index.

We choose telecommunication indices because the indices have been recorded for a reasonably long period.

For the R&D labor we use the number of scientists and engineers engaged in R&D up to 1997 (for UK up to 1996) in developed counties as a proxy.

	Period	Annual Growth
Japan Telecom	68-04	4.01%
Japan S&E in R&D	68-97	4.75%
UK Telecom	84-04	1.84%
UK S&E in R&D	84-96	1.04%
US Telecom	71-04	1.42%
NASDAQ	65-04	3.53%
US S&E in R&D	71-97	2.97%

The time periods covered in the table represent those reported in the original data sources, as complete as we can get.

The returns for equity are adjusted by the consumer price indices, as the returns and wages in the current paper are real returns and real wages.

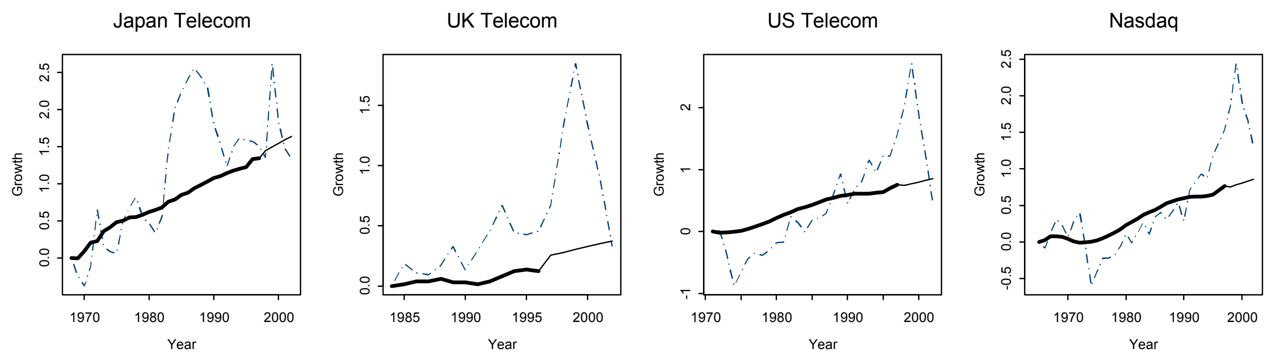


Figure 1:

The difference between public returns of R& D and private returns of R & D.

Jones and Williams (1998)

2.2.1 Cross-Sectional Implication

Cross-sectionally the model provides an explanation of an "empirical puzzle" observed for growth stocks.

A specific assumption on $\xi_j(t)$, the research effectiveness:

$$d\xi_j(t) = \sigma^2(b - c\xi_j(t))dt + \sigma\sqrt{\xi_j(t)}dW(t),$$

where $W(t)$ is a standard Brownian motion.

Suppose at time t, one observes the K largest firms in terms of the total market capitalization $M_j(t)$, and rank them such that $M_{(1)}(t) > M_{(2)}(t) > \cdots > M_{(K)}(t)$, where $M_{(1)}$ denotes the largest firm, $M_{(2)}$ the second largest firm etc.

In fact, we can show that when $\xi' s$ reach the steady state.

$$\log \frac{M_{(j)}(t)}{M_{(1)}(t)} = - \frac{1}{1-2b} \log j - \frac{2c}{1-2b} (\xi_{(j)}(t) - \xi_{(1)}(t)).$$

Now if the last term is negligible, then it would imply that a plot of log-market-capitalization of large cap growth firms versus the log-rank would simply display a linear pattern.

Using the probabilistic results in Kou and Kou (2003, 2004), it can be shown here that under suitable conditions the last term will converge to zero.

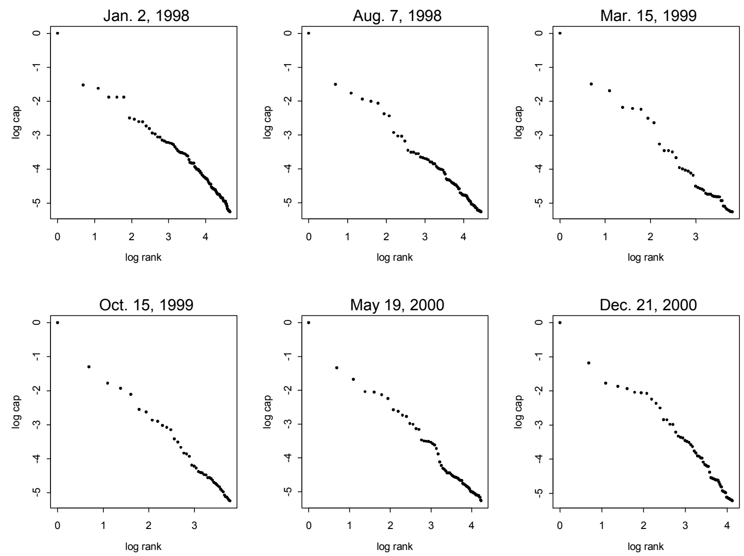


Figure 2: large biotechnology stocks

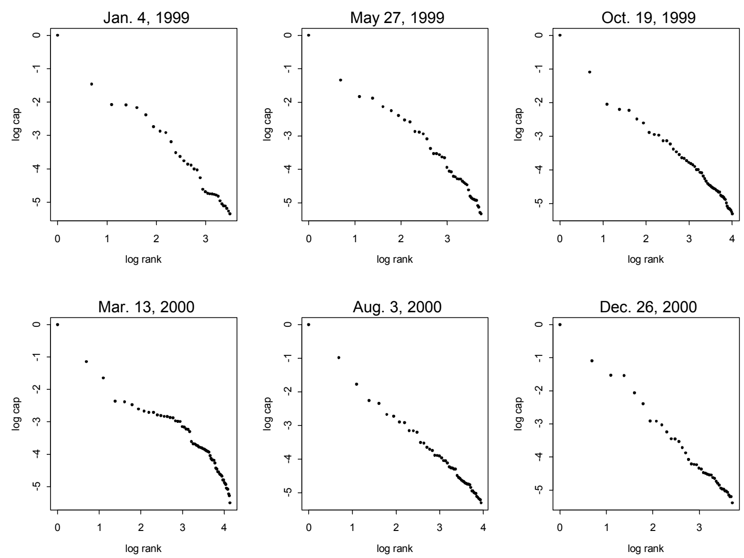


Figure 3: large internet stocks

In other words, the model provides an economic explanation of the empirically observed size distribution puzzle for growth stocks, which include not only internet stocks (as the Wall Street Journal article focused) but also biotechnology stocks. Also we achieve a better fit in terms of R^2.

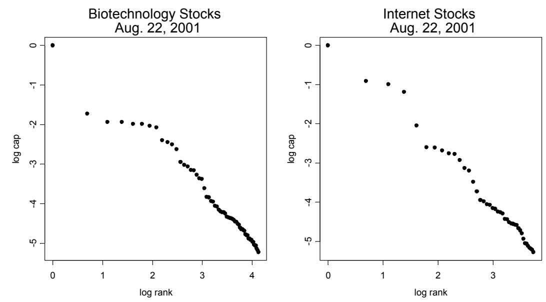


Figure 4: August 22, 2001.

A simple least square fitting appears to be very good; in fact, the R^2 statistics from the fitting of the 14 cross-sectional graphs in Figures 1-3 all exceed 94%.

It is also interesting to note that the almost linear pattern is present irrespective of market upturns and downturns.

One further point worth mentioning is that in the derivation we assumed that the ξ_j's are from the stationary distribution. Kou and Kou (2003, 2004) proved that a high volatility σ^2 will lead to a fast convergence to its stationary distribution.

Kijima's definition of "decay parameter."

This suggests that the assumption of stationarity is quite reasonable for growth stocks, thanks to their high volatility.

This also explains why the same should not be expected for ordinary stocks with low volatility.

Size distribution has a long history in economics and statistics; see, e.g., Pareto (1896), Yule (1924), Zipf (1949). Modern applications of size distribution in economics started from Simon (1955); see Ijiri and Simon (1977), Krugman (1996), and Gabaix (1999) for more discussion.

The convergence speed and the application of size distribution in studying growth stocks appear to be new in the size distribution literature.

3 Description of the Stochastic Endogenous Growth Model

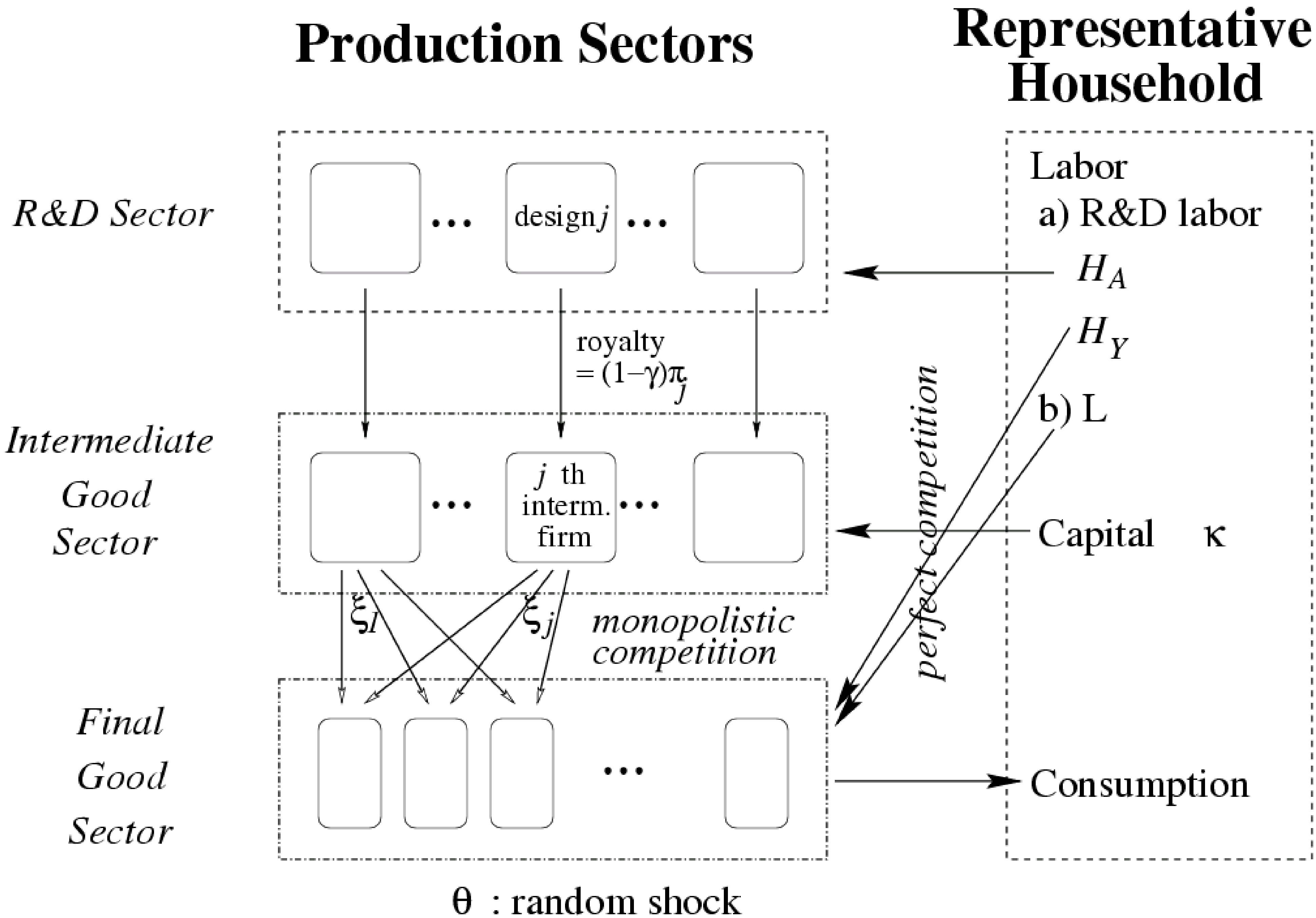
In this paper, we shall first extend Jones version of the Romer endogenous R&D model from a deterministic setting to a stochastic one, and then use it to study growth stocks.

To do this, as in Romer (1990), we assume that the whole economy consists of three sectors, the research sector, the intermediate-good sector, and the final-good sector.

The research sector generates new ideas that increases the variety of intermediate goods.

After purchasing the exclusive right to produce a specific intermediate good, the intermediate sector, as a monopolist, manufactures the intermediate good.

The final-good sector uses the intermediate goods to produce the final output.



Production Sectors
Representative Household
R&D Sector
design j
Labor
a) R&D labor
H_A
H_Y
royalty
$= (1-\gamma)\pi_j$
b) L
Intermediate Good Sector
j th interm. firm
Capital κ
perfect competition
ξ_1
ξ_j
monopolistic competition
Final Good Sector
Consumption
θ : random shock

4 Discussion

The contribution of the paper is to provide a model for

- stochastic endogenous growth
- growth stocks

We end the discussion by borrowing the wisdom in a quotation by Charles Dickens.

"It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us"

— A Tale of Two Cities

5 The Final-good Sector

5.0.2 The Production Function

Similar to the classical Solow growth model, the production function is assumed to be of the Cobb-Doglass form with random shocks:

$$Y(t) = \theta^{\alpha+\beta}(t) H_Y^{\alpha}(t) L^{\beta}(t) \sum_{j=1}^{A(t)} \left(\xi_j(t) x_j\right)^{1-\alpha-\beta}, \tag{2}$$

$$0 < \alpha, \beta < 1, \ \alpha + \beta < 1,$$

Here $H_Y(t)$ and $L(t)$ are the R&D labor and non R&D labor needed for the final production, respectively.

x_j are quantities of different intermediate goods which have to be purchased from the firms in the intermediate sector.

$A(t)$ measures the number of intermediate goods available for the final production; and random shocks come from $\theta(t)$ and $\xi_j(t)$.

The first factor $\theta(t)$ measures the production shock to the overall economy. The only assumption on $\theta(t)$ is that $\theta(t)$ is a nonnegative mean reverting process. No functional form of $\theta(t)$ is assumed in the model.

Note that if $\theta(t)$ is a nonnegative mean reverting process, then so is $\theta^{\alpha+\beta}(t)$. Therefore, it does not matter whether we postulate the

mean reverting assumption on $\theta(t)$ or $\theta^{\alpha+\beta}(t)$.

The second factor $H_Y(t)$ in the production function represents the R&D labor needed for the final production of Y; the rest of the labor force in the final good production is represented by $L(t)$.

We assume that the total available R&D labor in the society $H(t)$ is divided among the final production sector $H_Y(t)$ and the research sector $H_A(t)$; i.e. $H(t) = H_Y(t) + H_A(t)$.

For simplicity, we assume that $L(t)$ and $H(t)$ are exogenous variables (otherwise, we have to study the population growth, educational effects, etc., which detours from the basic idea of the paper).

However, we shall see in the later sections that in equilibrium how to divide $H(t)$ into $H_Y(t)$ and $H_A(t)$ is determined endogenously in the model.

The summation index $A(t)$ measures the total number of intermediate products available. The final-good sector takes this number as given.

Following Romer (1990), we assume that the contribution of research is to increase $A(t)$, i.e. increase the variety of products available in the society.

Inside the summation of $Y(t)$, the stochastic process $\xi_j(t)$ measures the effectiveness of the jth product x_j towards the final production, hence also the effectiveness of the research activity in the research sector, which invents the idea of producing x_j.

No functional form of $\xi_j(t)$ is assumed now. The only assumptions on $\xi_j(t)$ are:

(1) For $j \geq 1$, $\xi_j(t)$ are independently and identically distributed with $\xi_j(0)$ being equal for all $j \geq 1$;

(2) $\xi_j(t)$ is a nonnegative mean reverting process;

(3) $\xi_j(t)$ has a high volatility, reflecting the fact that the research effectiveness is very unpredictable.

Furthermore, the two random shocks, the economy-wise production shock $\theta(t)$ and the effectiveness $\xi_j(t)$ of the jth product, are assumed to be independent of each other.

Since the market for the final good sector is assumed to be competitive, the wage w_Y for H_Y, the wage w_L for L, and demand curve for x_j can be determined through competition and first order conditions.

In equilibrium, since the production of final good is fully competitive, the profit for the final good firms after the wages and capital costs is zero.

5.1 Intermediate-Good Sector

Each firm in the intermediate-good sector first purchases a patent (or design) from the research sector. Afterwards, it acts as a monopolist in the production of the particular intermediate good and then sells it to the final-good sector.

Each firm is assumed to produce only one intermediate good, as the patent protection allows only one firm to manufacture one intermediate good.

Romer (1990) has a detailed discussion on why monopoly is essential in view of the nonrivalry of ideas.

5.1.1 Monopolistic Price and Profit

Because of the monopoly, at any time t, with the demand function $p_j(x_j)$ from the final-good sector, the intermediate firm j will choose an optimal quantity x_j that maximizes the profit:

$$\max_{x_j} \pi_j(t), \quad \pi_j(t) = p_j(x_j)x_j - \eta x_j,$$

where $p_j(x_j)$ is the monoply price, and η is the manufacture cost per item.

Note that unlike the final product firms, which are fully competitive and have zero profit, there is a profit due to the monopoly.

Again, unlike the deterministic Romer model, in the presence of random shocks, the monopolistic profit $\pi_j(t)$ will be stochastic and different across different intermediate firms.

5.1.2 Financing the Intermediate-Good Sector and the Monopolistic Competition

To produce the jth intermediate good, the jth firm in the intermediate sector needs to get an exclusive right to manufacture the good, and a series of capital investments.

Note that in the Romer model the financing part is absent.

To finance the cost, an intermediate firm will issue shares of equity; thus the shareholders of the jth firm will supply the capital, while the research sector, which holds the patents, will transfer the right to manufacture at the cost of a royalty.

We shall assume that the intermediate sector behaves as monopolistic competition.

It is worth pointing out that the meaning of monopolistic competition here is different from that in Romer (1990).

In Romer's model monopolistic competition means that all the profits from the intermediate sector are extracted by the patent holders to compensate the researchers for the time they spend in inventing new designs.

In our model, monopolistic competition means:

(1) All the profits from the intermediate sector are distributed both to the research sector and the equity shareholders of intermediate firms.

(2) The issuing of the equity will be done in a competitive, free entry capital market, so that the value of the equity shares will be determined by the free market.

Therefore, although there is monopoly, all profits compensate the input factors, i.e. the research activities as well as the capital needed to transform ideas into intermediate goods.

5.2 Research Sector

The research sector generates designs for new intermediate goods. The designs are sold to the intermediate sector immediately after their invention, in return for a royalty.

5.2.1 The Number of Designs $A(t)$

Following Jones (1995a),

$$\dot{A}(t) = \frac{dA(t)}{dt} = \bar{\delta} H_A(t), \tag{3}$$

where

$$\bar{\delta} = \delta H_A^{\lambda-1}(t) A(t)^{\varphi},\ 0 < \lambda < 1, \quad \varphi < 1, \tag{4}$$

λ is the coefficient of labor contribution to research and technology growth, and $H_A(t)$ is the R&D labor devoted to the research sector. (Note that the total R&D labor is divided into the research sector $H_A(t)$ and the final product sector $H_Y(t)$.)

Here the constraint $\lambda \in (0,1)$ reflects the possibility of research duplication: due to potential duplication, research productivity falls as more labor enters the research sector. The coefficient φ attempts to capture the factor of positive knowledge spillover.

Note that in Romer's original model $\varphi = 1$. Jones (1995a, 1995b) gave a detailed discussion on why the forms $\varphi < 1$ and $\lambda \in (0,1)$ are appropriate from both theoretical and empirical aspects.

Note also that although the research effort $A(t)$ may be deterministic (as we shall see in later sections that $H_A(t)$ may be deterministic along the balanced growth path), the effectiveness of research measured by $\xi_j(t)$ is always random. Therefore, the whole research process is stochastic.

6 Equilibrium Equity and Patent Prices

To investigate the equilibrium equity value $M_j(t)$ of share holders and the equilibrium patent price $P_A^{(j)}(t)$, the royalty to the research sector is assumed to be a fixed proportion $(1-\gamma_j)$ of the profit with γ_j to be determined later; more precisely, the payment of the royalty is a (random) continuous cash flow $(1-\gamma_j)\pi_j(t)$ at any time t.

For a theoretical justification of the optimality of the fixed-fraction royalty, see, for example, the discussion and references in Asmati and Pfleiderer (1994).

To determine the equity value, we shall solve the problem under a rational expectation framework (Lucas, 1978).

$$E\left\{\int_t^\infty e^{-\rho(s-t)}U(c(s))ds|\mathcal{F}_t\right\}.$$

To obtain economic intuition without being blurred by technicalities and for simplicity, we shall assume a particular form of utility function, i.e. $U(c_t) = \log(c_t)$, just as we have assumed a particular form of the production function.

The logarithm utility function has been widely used in many previous works on economic growth theory mainly because of its analytical tractability; see, for example, Brock and Mirman (1972), and Long and Plosser (1983).

7 Equilibrium R&D Labor Allocation and the Balanced Growth Path

We will first investigate what the values of $H_Y(t)$ and $H_A(t)$ are in equilibrium. It will then be used to determine the rate of economic growth along the balanced growth path.

The equilibrium R&D labor allocation is studied by matching the wages for the R&D labor in the research and final good production sectors.

7.1 Matching the Wages

In equilibrium the wage w_Y in the final good sector must match the average wage, $\bar{w}_A$, in the research sector:

$$w_Y = \bar{w}_A,$$

where

$$\bar{w}_A = \frac{1}{A(t)} \sum_{j=1}^{A(t)} w_A^{(j)}(t) = \bar{\delta} \frac{1}{A(t)} \sum_{j=1}^{A(t)} P_A^{(j)}(t). \tag{5}$$

Note that both w_Y and $\bar{w}_A$ are stochastic processes.

There is an important difference between the deterministic endogenous growth model in Romer (1990) and the stochastic model here. In Romer's model, since there is no randomness, $P_A^{(j)}(t)$ are the same for all $j \geq 1$, whereas here in the presence of stochastic terms, $P_A^{(j)}(t)$ (and thus $w_A^{(j)}(t)$) are not only different for different $j \geq 1$, but stochastic as well.

Consequently, it is impossible to match the wage w_Y with individual $w_A^{(j)}(t)$. However, it is possible to match the wage w_Y in the production sector with the *average* wage $\bar{w}_A$ in the research sector.

Therefore, matching the wages together with

$$\dot{A}(t) = \frac{dA(t)}{dt} = \bar{\delta} H_A(t) = \delta H_A^{\lambda}(t) A(t)^{\varphi}$$

and

$$H_Y(t) + H_A(t) = H(t)$$

describes the equilibrium R&D labor allocation in the whole economy.

The total R&D labor force $H(t)$ in the population is assumed to be a deterministic geometric process with growth rate $\frac{\dot{H}(t)}{H(t)} = g_H$.

7.2 Equilibrium R&D Labor Allocation along the Balanced Growth Path

Result 1:

$$\frac{H_A(t)}{H_Y(t)} = \frac{1}{\rho}(1-\alpha-\beta)(1-\gamma)(\frac{\alpha+\beta}{\alpha})\frac{\dot{A}(t)}{A(t)}. \tag{6}$$

In general, it is difficult to analyze the whole economy and its growth at an arbitrarily given time, even if every process under consideration is deterministic.

Following the convention in economic growth literature, we shall focus on the implication of the model along the balanced growth path. In this way, the mathematics becomes tractable.

In a balanced growth path, $\frac{\dot{A}(t)}{A(t)} = g_A$ is a constant. Hence the fraction of R&D Labor in the research sector is also constant.

7.3 Discussion

The contribution of the paper is to provide a model for

- stochastic endogenous growth
- growth stocks

We end the discussion by borrowing the wisdom in a quotation by Charles Dickens, which is pertinent to the recent period of the boom and burst of growth stocks.

“It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us”

— A Tale of Two Cities