Backward SDEs with Jumps and Applications in Utility Optimization

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Recall the Black-Scholes Model

- Discounted risky asset price dynamic ($r = 0$) under the risk-neutral measure Q has no drift

$$dS_t = S_t \, \sigma_t dW_t$$

- Price of a European option $B = (S_T - K)^+$

$$V_t := E_t^Q [B] \, , \quad t \in [0, T]$$

  is a martingale under Q.

- Completeness of the model comes from the martingale representation property of the Brownian motion

$$dV_t = \theta^W \, dW = \theta^S \, dS \, , \quad \theta^S = (S\sigma)^{-1} \theta^W$$

- $\rightsquigarrow$ Perfect Hedging = Replication

How to Model Additional Risky Events?

- Introduce ...
  - Point processes $\rightsquigarrow$ unpredictable events
  - Marked point processes $\rightsquigarrow$ event magnitudes
  - Multivariate processes $\rightsquigarrow$ multiple event types
- Examples
  - Credit/default events in reduced form models
  - Occurrence of catastrophes
  - Aggregated insurance losses
  - Regime switching of economic state variables
- But: Model becomes incomplete
  $\rightsquigarrow$ How to value and hedge ?

Questions

- Could we have a Weak Representation Property, s.t.

$$V_t + \int_t^T \theta \, dW + \int_t^T U_s \, dM = \quad B + \int_t^T f_s(U_s) \, ds \quad ?$$

Valuation + Hedging + Residual = Claim + Resid.Risk premium

- Example: Poisson process $N \rightsquigarrow M = N - \int \lambda_s \, ds$ is martingale orthogonal to W, and W and M span all uncertainty.
  But how to deal with (non-linear) $\int f_s ds$?
- How to assign a risk-averse premium f to the hedging error?
- Intuition: f should be positive and convex
- Wanted: One solution (V, θ, U) for any given (B, f)
- Are there asymptotics which lead to simpler approx. solutions?

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Outline

1. Mathematical Tools - Backward SDE's with Random Measures
2. Exponential Utility Maximization
3. Utility Indifference Hedging and Valuation
4. Markovian Example with PDE solutions

Tools for Jumps - Random Measures

- Random measure, integer-valued, $E := (\mathbf{R} \setminus \{0\})$,

$$\mu(\omega, dt, de) = \sum_s \delta_{(s,\beta(\omega,s))}(dt, de) 1_D(\omega, s)$$

$$\mu : \Omega \times [0, T] \times (\mathbf{R} \setminus \{0\}) \rightarrow \bar{\mathbf{N}}$$

- Predictable compensator $\nu(\omega, dt, de)$

$$\nu : \Omega \times [0, T] \times (\mathbf{R} \setminus \{0\}) \rightarrow \bar{\mathbf{R}}$$

- Example: Marked point process $\sum_{\{i: T^i \leq ty\}} U_i$, jump intensity λ , jump sizes i.i.d.

$$\mu(dt, de) = \sum_i \delta_{(T^i, U_i)}(dt, de) 1_{\{T^i | i \in \mathbf{N}\}}(\omega, s)$$

$$\nu(dt, de) = \mathrm{density}_U(de) \lambda dt$$

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- Predictable compensator $\nu(\omega, dt, de)$

$$\nu : \Omega \times [0, T] \times (\boldsymbol{R} \setminus \{0\}) \rightarrow \bar{\boldsymbol{R}}$$

- Example: Jumps process $\Delta X_t = X_t - X_{t-}$ of a semimartingale X

$$\mu(dt, de) = \sum_{s} \delta_{(s,\Delta X_s)}(dt, de) 1_{\{s:\Delta X \neq 0\}}(dt)$$

Tools: Weak Representation Property with Random Measures

- Brownian motion W, Integer values Random measure μ with compensator $\mu(dt, de) = \zeta(\omega, t, e)\lambda(de)dt$ with $\lambda(E) < \infty$.
- Assumption: W and $\widetilde{\mu} := \widetilde{\mu} - \nu$ satisfy the weak predicatable representation property

$$M_t = M_O + \int_0^t \theta_s dW + \int_0^t \int_E U(s, e)\widetilde{\mu}(ds, de)$$

- Example: Brownian motion and independent Poisson process
  Example: Gaussian part and jumps part of a Levy process

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- Example: Brownian motion and independent multivariate marked point process
  Example: previous examples under a change of measure

Main Tool - BSDE's with Random Measures

- Data given: Claim B (bounded) and generator f
- Backward stochastic differential equation (BSDE)

$$
\begin{aligned}
Y_t &= B + \int_t^T f_s(Y_s, Z_s, U_s)\, ds \\
&\quad - \int_t^T Z_s\, dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(dt, de)
\end{aligned}
$$

- Solution is a triple of adapted processes (Y, Z, U)
- Construction: Fixed-point iteration in a L^2-Hilbert space
- This needs Lipschitz properties of the generator f, cf Barles, Buckdahn, Pardoux (1997).
- Lipschitz is too restrictive + square integrability not enough...
- But: Can find bounded solution by truncation+stopping.

Financial market model

- Continuous risky asset prices

$$dS = S\Sigma\,(\phi\,dt + dW)$$

- Simplify w.l.o.g to

$$dS = \phi\,dt + dW = d\widehat{W}\,.$$

- Bounded market price of risk ϕ
- Claim $B \in L^\infty$ can depend on evolution of S and μ.
- Trading strategies $\theta \in \Theta$ of bounded mean oscillation
- Investor's objective: Maximize expected exponential utility of wealth from time T.
- $U(x) := -e^{-\alpha x}$, $\alpha > 0$: Exponential utility function

Utility Maximization by BSDE's

- The solution (Y^B, Z^B, U^B) to the BSDE

$$Y_t = B + \int_t^T \left(-\frac{|\phi|^2}{2\alpha} + \int_E \left(\frac{\exp(\alpha U_s(e))}{\alpha} - \frac{1}{\alpha} - U_s(e) \right) \zeta(t,e)\lambda(de \right.$$
$$- \int_t^T Z_s \, d\widehat{W} - \int_t^T \int_E U_s(e) \, \widetilde{\mu}(dt, de) \,, \quad t \in [0, T].$$

  under the minimal martingale EMM $\widehat{P} = \mathcal{E}(-\int \phi \, dW)$

- Solves the utility maximization problem

$$-\exp(\alpha Y_t^B) = \operatorname*{ess\,sup}_{\theta \in \Theta} E_t^P \left[-\exp\left(-\alpha \left(+ \int_t^T \theta \, d\widehat{W} - B \right) \right) \right]$$
$$Z^B + \frac{\phi}{\alpha} = \theta^B$$

Solution to the Dual Problem by BSDE's

- The dual problem consists in finding the EMM

$$Q^{E,B} = \mathrm{argmax}_{Q \in \boldsymbol{P}_f} \left\{ \alpha E^Q[B] - H(Q|P) \right\} .$$

- For $B = 0$, this is the minimal entropy measure
- The density process of $Q^{E,B}$ is

$$Z^{E,B} = \mathcal{E}\left(-\int \phi \, dW + \int \int_E \exp(\alpha U_s^B(e)) - 1 \; \widetilde{\mu}(ds, de) \right)_t$$

Utility Indifference Valuation and Hedging

$U(x) := -e^{-\alpha x}$, $\alpha > 0$: Exponential utility function

- Utility indifference value $\pi(B;\alpha)$:= solution of

$$\max_{\theta} E\left[U\Big(x + \int \theta dS\Big)\right] \stackrel{!}{=} \max_{\xi} E\left[U\Big(x + \pi + \int \xi dS - B\Big)\right]$$

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- Utility indifference hedging strategy
  $\psi(B;\alpha) := \theta^B - \theta^0 := \xi^{\max} - \theta^{\max}$
- Optimal strategy θ^B with liability B equals
  ordinary optimal strategy θ^0 + hedging strategy $\psi(B)$

General Valuation Properties

Assumptions: B sufficiently integrable, no duality gap, well-posedness: $\boldsymbol{P}_f := \boldsymbol{P}_e \cap \{Q |\, H(Q|P) < \infty\} \neq \emptyset$

- Primal problem:
  $\theta^B \leftarrow \sup_{\theta \in \Theta} E^P \left[-\exp(-\alpha(x + (\int \theta dS)_T - B))\right]$
- Dual problem:
  $Q^{E,B} \leftarrow \sup_{Q \in \boldsymbol{P}_e} \left\{E^Q[B] - \frac{1}{\alpha}(H(Q|P) - H(Q^0|P))\right\}$
- Duality: optima exists and are in one-to-one correspondence

$$\frac{dQ^{E,B}}{dP} = \exp\left(-\alpha(\text{const} + (\int \theta^B dS)_T - B)\right)$$

- Problem: Solution to dual problem is typically not known.

Indifference Solutions by BSDE's

- The solution (Y^E, Z^E, U^E) to the BSDE (under $Q^{E,0}$)

$$\begin{aligned} Y_t &= B + \int_t^T \int_E \left(\frac{\exp(\alpha U_s(e))}{\alpha} - \frac{1}{\alpha} - U_s(e) \right) \zeta^E(t,e)\lambda(de)\, dt \\ &\quad - \int_t^T Z_s \, d\widehat{W} - \int_t^T \int_E U_s(e)\, \widetilde{\mu}^E(dt, de)\,, \quad t \in [0, T]. \end{aligned}$$

- solves the (dynamic) indifference valuation and hedging problem

$$\begin{aligned} \pi_t^\alpha &= Y_t^E \\ \psi_t^\alpha &= Z_t^E \end{aligned}$$

- and satisfies $\pi^\alpha = Y^B - Y^0$.

Solution for Vanishing Risk Aversion by BSDE's

- The BSDE under $Q^{E,0}$

$$Y_t^{E,0} = B - \int_t^T Z_s^{E,0}\, d\widehat{W} - \int_t^T \int_E U_s^{E,0}(e)\, \widetilde{\mu}^E(dt, de)\,, \quad t \in [0, T].$$

- provides the risk-minimizing strategy $Z^{E,0}$ and valuation $Y^{E,0}$
- This is the limit of the indifference valuation and hedging $(\pi^\alpha, \psi^\alpha)$ in that

$$E^{Q^{E,0}}\Bigg[\sup_{u\in[0,T]} |\pi^\alpha - Y^{E,0}|^2 + \int_0^T |\psi^\alpha - Z^{E,0}|^2\, ds + \int_{[0,T]\times E} |U^{E,\alpha} - U^{E,0}|^2 \nu^E(ds, de)\Bigg] \le \alpha^2 \mathrm{const}\,,$$

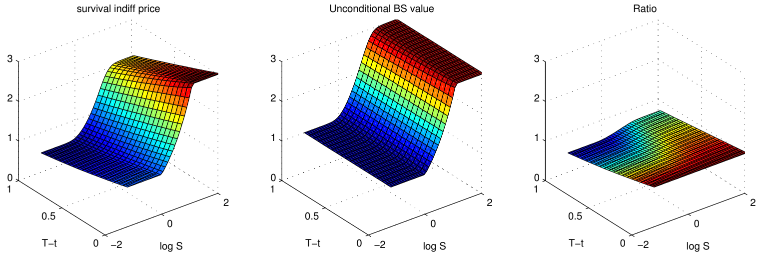
A Markovian Example with PDE solution

- Example: Event-conditional Index-linked Payoff for instance an index-linked life insurance, or a CAT-conditional stock-index linked payoff
- Stock-index: Black-Scholes diffusion

$$\frac{dS_t}{S_t} = \gamma(t, S_t)\, dt + \sigma(t, S_t)\, dW_t$$

- Poisson processes (N_t) with intensity $\lambda(t)$
- Contingent Claim of Knock-out/in-type, e.g.

$$H = \begin{cases} f(S_T) & : \quad N_T = 0 \\ 0 & : \quad \text{otherwise} \end{cases}$$



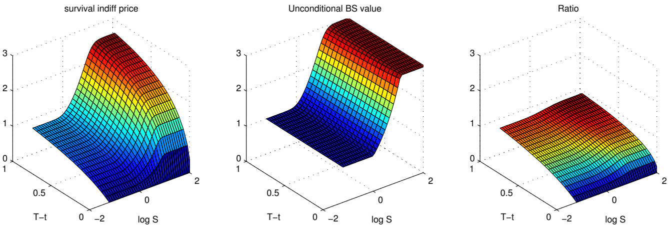
- non-linear PDE $v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} = -\lambda(t)\frac{1}{\alpha}(e^{\alpha(0-v)} - 1)$

$$v(T,x) = f(X), \quad f(X) = 1 + (S_T - K_1)^+ \wedge K_2$$

- ... yields indifference price and hedging strategy

$$\pi(H) = v(0, S_0)$$

$$\psi(H)(t, S_t, N_{t-}) = \begin{cases} v_x(t, S_t) & \text{if } N_{t-} = 0 \\ 0 & \text{otherwise} \end{cases}$$



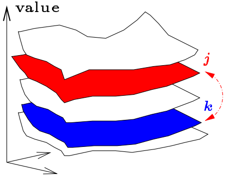
- non-linear PDE
$$\begin{aligned} v_t + \tfrac{1}{2}\sigma^2 x^2 v_{xx} &= -\lambda(t)\tfrac{1}{\alpha}\left(e^{\alpha(BS(f)-v)} - 1\right) \\ v(T,x) &= 0 \end{aligned}$$
- … yields indifference price and hedging strategy

$$\begin{aligned} \pi(H) &= v\,(0, S_0) \\ \psi(H)(t, S_t, N_{t-}) &= \begin{cases} v_x(t, S_t) & \text{if } N_{t-} = 0 \\ \mathrm{BSdelta}(f) & \text{otherwise} \end{cases} \end{aligned}$$

Utility Indifference - General Markovian Solution

- The Solution

$$\pi(H) = v(0, S_0, \eta_0)$$
$$\psi(H)(t, S_t, \eta_{t-}) = v_x(t, S_t, \eta_{t-})$$



- is described by the solution $v(t, x, k) = v^k(t, x)$ (and w) to
  - PDE system(s)

$$v_t^k + \frac{1}{2}\sigma^2 x^2 v_{xx}^k = -\delta - \sum_{j \neq k} \lambda^{kj} e^{\alpha(w^j - w^k)} \frac{1}{\alpha} \left(e^{\alpha(v^j - v^k + f^{kj})} - 1 \right)$$
$$v^k(T, x) = h(x, k)$$

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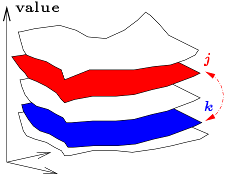

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$$w_t^k + \frac{1}{2}\sigma^2 x^2 w_{xx}^k = \frac{1}{2\alpha}||\frac{\gamma}{\sigma}||^2 - \sum_{j \neq k} \lambda^{kj} \frac{1}{\alpha}(e^{\alpha(w^j - w^k)} - 1)$$
$$w^k(T, x) = 0$$

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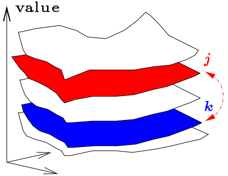

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F i n e

Thank you !

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