Sensitivity analysis of utility based prices and risk-tolerance wealth processes

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Based on a paper with Mihai Sirbu from Columbia University
The prices of non replicable derivative securities depend on many factors:

1. risk-preferences of the investor:
   (a) reference probability measure $\mathbb{P}$
   (b) utility function $U = U(x)$

2. current portfolio of the investor

3. trading volume in the derivatives

Goal: study the dependence of prices on trading volume.
There are $d + 1$ traded or liquid assets:

1. a savings account with zero interest rate.

2. $d$ stocks. The price process $S$ of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

$\mathcal{Q}$: the family of local martingale measures for $S$.

Assumption (No Arbitrage)

$\mathcal{Q} \neq \emptyset$
Contingent claims

Consider a family of $m$ non-traded or illiquid European contingent claims with

1. maturity $T$

2. payment functions $f = (f_i)_{1 \leq i \leq m}$.

Assumption No nonzero portfolio of $f$ is replicable:

$$\langle q, f \rangle = \sum_{i=1}^{m} q_i f_i \text{ is replicable } \iff q = 0$$
Pricing problem

Question  What is the (marginal) price \( p = (p_i)_{1 \leq i \leq m} \) of the contingent claims \( f \)?

Intuitive Definition  The marginal price \( p \) for the contingent claims \( f \) is the threshold such that given the chance to buy or sell at \( p^{\text{trade}} \) an investor will

\[
\begin{align*}
\text{buy at } & p^{\text{trade}} < p & \text{sell at } & p^{\text{trade}} > p \\
\uparrow & \\
\text{do nothing at } & p^{\text{trade}} = p
\end{align*}
\]
Consider an investor with the portfolio \((x, q)\), whose preferences are modeled by a utility function \(U\):

1. \(U : (0, \infty) \rightarrow \mathbb{R}\), strictly increasing and strictly concave

2. The Inada conditions hold true:

\[
U'(0) = \infty \quad U'(\infty) = 0
\]
Problem of optimal investment

The goal of the investor is to maximize the expected utility of terminal wealth:

\[ u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)], \]

where \( \mathcal{X}(x) \) is the set of strategies with initial wealth \( x \).

Order structure: a portfolio \((x, q)\) is better than a portfolio \((x', q')\) if \( u(x, q) \geq u(x', q') \).
Marginal utility based price

**Definition**  
A marginal utility based price for the claims \( f \) given the portfolio \((x, q)\) is a vector \( p(x, q) \) such that  
\[
    u(x, q) \geq u(x', q')
\]
for any pair \((x', q')\) satisfying  
\[
    x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.
\]

In other words, given the portfolio \((x, q)\) the investor **will not trade** the options at \( p(x, q) \).
Computation of \( p(x) = p(x, 0) \)

Define the conjugate function

\[
V(y) = \max_{x > 0} [U(x) - xy], \quad y > 0.
\]

and consider the following **dual** optimization problem:

\[
v(y) = \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ V \left( y \left( \frac{dQ}{dP} \right) \right) \right], \quad y > 0
\]

\( \mathcal{Q}(y) : \) the **minimal martingale measure** for \( y \).
Computation of $p(x) = p(x, 0)$

Mark Davis gave heuristic arguments to show that if $y$ corresponds to $x$ in the sense that

$$x = -v'(y) \iff y = u'(x)$$

then

$$p(x) = \mathbb{E}_{\mathcal{Q}(y)}[f].$$

The precise mathematical results are given in a joint paper with Julien Hugonnier and Walter Schachermayer.
Computation of $p(x) = p(x, 0)$

Theorem (Hugonnier,K.,Schachermayer) Let $x > 0$, $y = u'(x)$ and $X$ be a non-negative wealth process. The following conditions are equivalent:

1. $p(x)$ is unique for any $f$ such that

   \[ |f| \leq K (1 + X_T) \quad \text{for some } K > 0 \]

2. $Q(y)$ exists and $X$ is a martingale under $Q(y)$.

Moreover, in this case $p(x) = \mathbb{E}_{Q(y)}[f]$. 
Trading problem

Assume that the investor can trade the claims at the initial time at the price $p^{\text{trade}}$.

**Question** What quantity $q = q(p^{\text{trade}})$ the investor should trade (buy or sell) at the price $p^{\text{trade}}$?

Using the marginal utility based prices $p(x, q)$ we can compute the optimal quantity from the “equilibrium” condition:

$$p^{\text{trade}} = p(x - qp^{\text{trade}}, q)$$
Sensitivity analysis of utility based prices

Main difficulty : \( p(x, q) \) is hard to compute except for the case \( q = 0 \).

Linear approximation for “small” \( \Delta x \) and \( q \):

\[
p(x + \Delta x, q) \approx p(x) + p'(x) \Delta x + D(x)q,
\]

where \( p'(x) \) is the derivative of \( p(x) \) and

\[
D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad 1 \leq i, j \leq m.
\]
Quantitative questions

Question (Quantitative) How to compute $p'(x)$ and $D(x)$?

Closely related publications:

J. Kallsen (02) : formula for $D(x)$ for general semimartingale model but in a different framework of local utility maximization.

V. Henderson (02) : formula for $D(x)$ in the case of a Black-Scholes type model with basis risk and for power utility functions.
Qualitative questions

Question (Qualitative) When the following (desirable) properties hold true for any family of contingent claims $f$?

1. The marginal utility based price $p(x) = p(x, 0)$ does not depend (locally) on $x$, that is,

   $p'(x) = 0$

2. The sensitivity matrix $D(x)$ has full rank

3. The sensitivity matrix $D(x)$ is symmetric
Qualitative questions

4. The sensitivity matrix $D(x)$ is negative semi-definite:
   $$\langle q, D(x)q \rangle \leq 0.$$ 

5. **Stability** of the linear approximation: for any $p^{trade}$ the linear approximation to the “equilibrium” equation:
   $$p^{trade} = p(x - qp^{trade}, q)$$

that is,
   $$p^{trade} \approx p(x) - p'(x)qp^{trade} + D(x)q$$

has the “correct” solution.
Risk-tolerance wealth process

Definition (K., Sirbu) A maximal wealth process $R(x)$ is called the risk-tolerance wealth process if

$$R_T(x) = -\frac{U'(\hat{X}_T(x))}{U''(\hat{X}_T(x))},$$

where $\hat{X}(x)$ is the optimal solution of

$$u(x) := u(x, 0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$
Risk-tolerance wealth process

Some properties of $R(x)$ (if it exists):

1. Initial value:

   $$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

   $$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{\widetilde{X}(x + \Delta x) - \widetilde{X}(x)}{\Delta x}.$$
Main qualitative result

Recall \( p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q \).

Theorem (K., Sirbu) The following assertions are equivalent:

1. The risk-tolerance wealth process \( R(x) \) exists.
2. \( p'(x) = 0 \) for any \( f \).
3. \( D(x) \) is symmetric for any \( f \).
4. \( D(x) \) has full rank for any (non-replicable) \( f \).
5. \( D(x) \) is negative semidefinite for any \( f \).
Existence of $R(x)$

Recall that $Q(y)$ is the minimal martingale measure (the solution to the dual problem) for $y$.

**Theorem (K., Sirbu)** The following assertions are equivalent:

1. $R(x)$ exists.

2. $\frac{d}{dy} Q(y) = 0$ at $y = u'(x)$.

In particular, $R(x)$ exists for any $x > 0$ if and only if $Q(y)$ is the same for all $y$. 
Second order stochastic dominance

**Definition**  If $\xi$ and $\eta$ are nonnegative random variables, then $\xi \succeq_2 \eta$ if

$$\int_0^t \mathbb{P}(\xi \geq x) \, dx \geq \int_0^t \mathbb{P}(\eta \geq x) \, dx, \quad t \geq 0.$$  

We have that $\xi \succeq_2 \eta$ iff

$$\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$$  

for any convex and decreasing function $W$.  

Existence of $R(x)$

Case 1: a utility function $U$ is arbitrary.

Theorem (K., Sirbu) The following assertions are equivalent:

1. $R(x)$ exists for any $x > 0$ and any utility function $U$.

2. There exists a unique $\hat{Q} \in Q$ such that

$$\frac{d\hat{Q}}{dP} \geq 2 \frac{dQ}{dP} \forall Q \in Q.$$
Existence of $R(x)$

Case 2: a financial model is arbitrary.

Theorem (K., Sirbu) The following assertions are equivalent:

1. $R(x)$ exists for any $x > 0$ and any financial model.

2. The utility function $U$ is

   (a) the power utility: $U(x) = (x^\alpha - 1)/\alpha$, $\alpha < 1$, if $x \in (0, \infty)$;

   (b) the exponential utility: $U(x) = -\exp(-\gamma x)$, $\gamma > 0$, if $x \in (-\infty, \infty)$. 

Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a numéraire and denote

$$f^R = f R_0(x)/R(x) : \text{discounted contingent claims}$$

$$X^R = X R_0(x)/R(x) : \text{discounted wealth processes}$$

$$\hat{Q}^R : \text{the martingale measure for } X^R :$$

$$\frac{d\hat{Q}^R}{d\hat{Q}} = \frac{R_T(x)}{R_0(x)}$$
Computation of \( D(x) \)

Consider the Kunita-Watanabe decomposition:

\[
P_t^R = \mathbb{E}_{Q^R} \left[ f^R | \mathcal{F}_t \right] = M_t + N_t, \quad N_0 = 0,
\]

where

1. \( M \) is \( R(x)/R_0(x) \)-discounted wealth process.
   Interpretation: \textit{hedging process}.

2. \( N \) is a martingale under \( Q^R \) which is orthogonal to all \( R(x)/R_0(x) \)-discounted wealth processes.
   Interpretation: \textit{risk process}.
Computation of $D(x)$

Denote $a(x) := -xu''(x)/u'(x)$ the relative risk-aversion coefficient of

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Theorem (K., Sirbu) Assume that the risk-tolerance wealth process $R(x)$ exists. Then

$$D(x) = -\frac{a(x)}{x} \mathbb{E}_{QR} \left[ N_T N'_T \right]$$
Computation of $D(x)$ in practice

Inputs:

1. $\hat{Q}$. *Already implemented!*

2. $R(x)/R_0(x)$. Recall that

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\hat{X}(x + \Delta x) - \hat{X}(x)}{\Delta x}.$$ 

*Decide what to do with one penny!*

3. Relative risk-aversion coefficient $\alpha(x)$. *Deduce from mean-variance preferences*. In any case, this is just a number!
Model with basis risk

Traded asset: \[ dS_t = S_t (\mu dt + \sigma dW_t) \].

Non traded asset: \[ d\tilde{S} = (\tilde{\mu} dt + \tilde{\sigma} d\tilde{W}_t) \]

Denote by \[ \rho = \frac{d\tilde{W} dW}{dt} \] the correlation coefficient between \( S \) and \( \tilde{S} \). In practice, we want to chose \( S \) so that \[ \rho \approx 1. \]
Model with basis risk

Consider contingent claims \( f = f(\tilde{S}) \) whose payoffs are determined by \( \tilde{S} \) (maybe path dependent).

To compute \( D(x) \) assume (as an example) the following choices:

1. \( \hat{Q} \) is a martingale measure for \( \tilde{S} \).

2. \( R(x)/R_0(x) = 1 \)

Then

\[
D_{ij}(x) = -\frac{\alpha(x)}{x}(1 - \rho^2)\text{Cov}_{\hat{Q}}(f_i, f_j).
\]
Assumptions

Assumption  The financial model can be completed by an addition of a finite number of securities.

Assumption  There are strictly positive constants $c_1$ and $c_2$ such that $c_1 < -\frac{xU''(x)}{U'(x)} < c_2$, $x > 0$.

Assumption  There is a wealth process $X \geq 0$ such that $|f| \leq X_T$ and $X$ is a square integrable martingale under the minimal martingale measure $\mathbb{Q}(y)$.
Summary

- For non replicable contingent claims prices depend on the trading volume.
- The following conditions are equivalent:
  - Approximate utility based prices have nice qualitative properties
  - Risk-tolerance wealth processes exist.
- We need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numéraire.