Optimal Process Approximation:
Application to Delta Hedging and Technical Analysis

Bruno Dupire
Bloomberg L.P. NY

bdupire@bloomberg.net
• Financial data are recorded down to the tick transaction (virtually continuous time)

• How can we simplify this information into an exploitable format?
We mostly compare the classical "time based" approximation to the "move based" approximation.

The plan is the following:

- Theoretical results
- Delta Hedging
- Technical Analysis
Theoretical Results
We assume a continuous path \((t, X_t)\) which we want to approximate with a finite number of points \((t_i, X_{t_i})\)

A) Time based approximation: \((i \Delta t, X_{i \Delta t})\)

B) Move based approximation:

fix \(\alpha > 0\), \((\tau_i, X_{\tau_i})\)

where \(\tau_i \equiv \inf\{t > \tau_{i-1} : |X_t - X_{\tau_{i-1}}| \geq \alpha\}\)
To simplify the analysis, both approximations are piecewise constant, denoted $X^{\Delta t}$ and $X^{\alpha}$.
Convergence Properties

- $L^\infty$ convergence
  - Time based: no uniform $L^\infty$ convergence, just $E[\|X - X^{\Delta t}\|_{\infty}]$
  - Move based: by construction $\|X - X^\alpha\|_{\infty} \leq \alpha$

The knowledge of the string of ups and downs locates the path up to an error $\delta$.

- $L^1$, $L^2$ convergence for $X$ a Brownian motion:
  - Time based: $E[\|X - X^{\Delta t}\|_1] = \frac{T}{3} \sqrt{\Delta t} \sqrt{\frac{4}{\pi}}$  
    $E[\|X - X^{\Delta t}\|_2^2] = \frac{T}{2} \Delta t$
  - Move based: $E[\|X - X^\alpha\|_1] = \frac{T}{3} \alpha$  
    $E[\|X - X^\alpha\|_2^2] = \frac{T}{6} \alpha^2$
• Pathwise convergence of variations of all orders for move and time based approximations

• Furthermore if \( dY_t = \mu_t dt + \sigma_t dW_t \):

  - **Time based:** \( \lim_{\Delta t \to 0} \frac{1}{\sqrt{\Delta t}} (\langle Y^{\Delta t}_t \rangle - \langle Y_t \rangle) = \sqrt{2} \int_0^t \sigma_s^2 dB_s \)

  - **Move based:** \( \lim_{\alpha \to 0} \frac{1}{\alpha} (\langle Y^{\alpha}_t \rangle - \langle Y_t \rangle) = \sqrt{\frac{2}{3}} \int_0^t \sigma_s dB_s \)

  where \( B \) and \( W \) are two independent brownian motions
Furthermore if $\frac{dY_t}{Y_t} = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t$, for a European option with $\Gamma(t, Y_t)$:

- **Time based:** (Bertsimas, Kogan, Lo, 2000)
  \[
  \lim_{\Delta t \to 0} \frac{1}{\sqrt{\Delta t}} TE_t = \sqrt{\frac{1}{2}} \int_0^t \sigma_s^2 Y_s^2 \Gamma(s, Y_s) dB_s
  \]

- **Move based:**
  \[
  \lim_{\alpha \to 0} \frac{1}{\alpha} TE_t = \sqrt{\frac{1}{6}} \int_0^t \sigma_s Y_s \Gamma(s, Y_s) dB_s
  \]

$B$ and $W$ are two independent brownian motions
• $X$ continuous martingale, $X_0 = 0$

• $\tau_i$ date of $i^{th}$ crossing

• $L(\alpha, T) = \max\{i : \tau_i < T\} = n$

\[
X_{\tau_n}^2 = \sum_{i=1}^{n} (X_{\tau_i} - X_{\tau_{i-1}})^2 + 2 \sum_{i=1}^{n} X_{\tau_{i-1}} (X_{\tau_i} - X_{\tau_{i-1}})
\]

As $(X_{\tau_i} - X_{\tau_{i-1}})^2 = \alpha^2$, $\mathbb{E}[X_{\tau_n}^2] = \mathbb{E}[n] \alpha^2$

\[
\mathbb{E}[X_T^2] = \mathbb{E}[X_T^2 - X_{\tau_n}^2] + \alpha^2 \mathbb{E}[n]
\]

$\tau_n \leq T \leq \tau_{n+1} \Rightarrow \mathbb{E}[n] \alpha^2 = \mathbb{E}[X_{\tau_n}^2] \leq \mathbb{E}[X_T^2] \leq \mathbb{E}[X_{\tau_{n+1}}^2] = \mathbb{E}[(n + 1)] \alpha^2$

\[
\frac{\mathbb{E}[X_T^2]}{\alpha^2} - 1 \leq \mathbb{E}[L(\alpha, T)] \leq \frac{\mathbb{E}[X_T^2]}{\alpha^2}
\]
Alternate intervals of **bridges** and **meanders**. Sequence of ups and downs is the same when computed FWD or BWD (Indeed crossing times are different)
△ Hedging
Optimal Delta Hedging

• Optimal delta hedging is a complicated topic in the presence of transaction costs.

• Highly dependent on specific assumptions

• Simplified problem
  – Bachelier dynamics
  – Option with constant $\Gamma$: parabola
  – Expected time between two rehedges imposed
Stopping Times

\[ X(W, T) = W_T^2 \longrightarrow X(W_t, t) = \mathbb{E}_t[W_T^2] = W_t^2 + T - t \]

- Rehedge at stopping time \( \tau \): \( \delta P&L = W_T^2 - \tau \).
- As \( \mathbb{E}[W_T^2 - \tau] = 0 \) for \( \tau \) of finite expectation,

we aim at minimizing :

\[ \mathbb{E}[(W_T^2 - \tau)^2] \text{ subject to } \mathbb{E}[\tau] = 1 \]

or

\[ \min_{\tau} \sqrt{\frac{\mathbb{E}[(W_T^2 - \tau)^2]}{\mathbb{E}[\tau]}} = \text{Ratio} \]
First Attempts

• Time based

• Move based
Second Attempts

- Truncate the move based

- Follow the break-even points

Nothing beats the move based. Is move based optimal?
Optimality of the Move Based

- By 2 successive Ito formulas, transform the problem to Kurtosis minimization of $W_\tau$:

$$\mathbb{E}[(W_\tau^2 - \tau)^2] = \frac{2}{3}\mathbb{E}[W_\tau^4]$$

- As $\mathbb{E}[\tau] = \mathbb{E}[W_\tau^2] = 1$, our new problem is:
  find a density with fixed variance and minimum kurtosis. This is achieved by a binomial distribution.

- The only integrable $\tau$ s.t. $W_\tau$ follows a binomial density is

$$\inf\{t : |X_t| \geq \delta\}$$
Links with Business Time Delta Hedge

• Business Time delta hedge is based on the DDS time change

• Important to link P&L to realized variance

• Cross points of move based correspond to business activity

• The sequence $\tau_i^\alpha$ converges to the time change

• Assume you have a correct estimate of the time based vol and of the move based vol.
  Hedging prescription: decrease the variance parameter by $\alpha^2$ at each cross point.
Tracking Error Comparison
Technical Analysis
Technical Analysis

**Economic Theory**
Prices follow a random walk

**Technical Analysis**
Past may influence future

If Technical Analysis is valid, we need to

1. Identify patterns
2. Predict from patterns
Binary Strings: Binary Coding
Optimal Coding of Price History

- Many possible approaches (PCA, wavelet analysis, time based, move based)
- Move based: define set of levels, and each time a level different from the last one is hit, record it
Digitizing the Price History

- Information compression: condenses the recent past into a sequence of binary digits.

- Keeps track of sequences of levels that have been reached.
- Extracts the relevant moves regardless of the precise timing.

\[
\begin{align*}
\text{up} & \rightarrow 1 \\
\text{down} & \rightarrow 0 \\
\text{Sequence:} & \quad 101011000101
\end{align*}
\]
Move Based Coding Properties

- Theory of processes approximation: move-based coding
  - Captures relevant price movements
  - Filters out irrelevant periods
  - Linked to optimal $\Delta$ hedging
  - Good volatility estimate
  - Similar to Renko-Kagi coding
Deciphering the Financial Genome

- Akin to finding genes in a DNA sequence

\[
\ldots \text{GAATCGACTTGA}G\text{GGCTACG} \ldots
\]

Gene coding for a protein

\[
\ldots \text{0010111011101100100001110010} \ldots
\]

Pattern predicting a up

- In both cases, extracting meaning from noise
Binary Strings: Algorithm
Cracking the code

- Identify past strings similar to the current one
- Average their following bit (up/down) to get a trading signal
- Convert it into a trading rule
Distance between Strings

old ————>new

A = 0 1 1 0 0 1
B = 1 1 1 0 0 1
C = 0 1 1 0 0 0

• A: current string
• B and C differ from A by one bit
• B is closer to A than C is because most recent bits match
• Distance between strings X and Y

\[ d(X, Y) = \sum_{i=1}^{n} (x_i - y_i)^2 \alpha^{n-i} \]
Voting Scheme

• The bit to come is “voted” by each of the past strings.
• Each past string $X_i$ is followed by bit $\epsilon_i$

$$X_i \quad \epsilon_i$$

0 1 1 0 0 1 1

• Each past string $X_i$ votes $\epsilon_i$ with a weight that depends on $d_i = distance(x_i, A)$
• The result is the estimated probability of an up-move given by:

$$p = \frac{\sum e^{-\lambda d_i} \epsilon_i}{\sum e^{-\lambda d_i}}$$ (most similar strings have highest weights)
Trading strategy

The indicator $p$ (probability of an up move) defines the buy/sell rule

- If $p < L$: sell
- If $p > H$: buy

Filters out $L < p < H$

Exit rule: Unwind each position when next rung (up or down) is reached.
Binary Strings Predictive Power
Conditional Trees

Extra information regarding strategy results can be computed, such as a conditional tree into the future as seen today.
Most predictive patterns

Each past sequence votes for the next bit in the current sequence.
Bloomberg Implementation
Input Parameters

- **period**: defines the trading period

- **delta**: defines the levels. Levels are \( \frac{\text{delta}}{100} \times S \) apart from each other.

- **buy threshold, sell threshold** (H and P): if \( p > H \), buy, \( p < L \), sell

- **position size**: number of shares traded at each signal
Outputs

- Initial time series with entry points
- Probability of up move
- Net exposure
- P&L comparison of Binary Strings strategy and Buy&Hold
Binary Strings Summary

- Binary Strings scrutinizes historical data in search of violations of Efficient Market Hypothesis
- It is based on a very compact encoding
- It makes use of non parametric estimation
- It is opportunistic, not dogmatic: it does not favor trend following nor range trading per se; it just exploits the data optimally
Conclusion

• How to compress information depends on the objective

• Superiority of move based over time based

• Better delta hedging

• Good way to compact information for technical Analysis

• Better volatility estimates