Irreversible Investments under Dynamic Capacity Constraints

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Outline

- ▷ A problem of irreversible investment
- ▷ ...with a dynamic capacity constraint
- ▷ First order conditions of optimality and optimal stopping
- ▷ Construction of an optimal policy
- ▷ Illustration and application
- ▷ Extension to problems with costly disinvestment
- ▷ A representation problem and universal stopping signals for Dynkin games
- ▷ Conclusion

An irreversible investment problem

Control problem:

Maximize a firm's utility from investment

$$U(\theta) = \mathbb{E} \int_0^\infty e^{-\delta t} u(X_t, \theta_t)\, dt$$

where

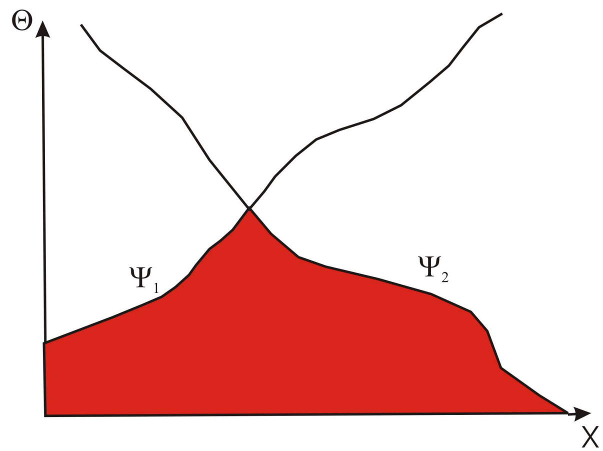
- ▷ $u(x, \vartheta)$ concave in ϑ: 'utility function'
- ▷ X geometric Brownian motion: 'state of the economy'
- ▷ $\theta_t \geq \underline{\vartheta} = \theta_0$ increasing, left-continuous, adapted control: 'cumulative investments'

Note:

- θ increasing $\leftrightsquigarrow$ investments are irreversible: factory etc.
- vast economic literature: Dixit & Pyndick, Scheinkman & Zariphopoulou, . . .

An irreversible investment problem

Solution by Kobila (1993):



$$\psi_1(\vartheta) = \inf\{x > 0 \mid \int_0^x u_\vartheta(y,\vartheta) y^{\gamma_1}\, dy > 0\}$$

$$\psi_2(\vartheta) = \sup\{x > 0 \mid \int_x^\infty u_\vartheta(y,\vartheta) y^{\gamma_2}\, dy > 0\}$$

Explicit construction of optimal policy based on Hamilton-Jacobi-Bellman equation:

$$\max\{\mathscr{G} v + u, v_\vartheta\} = 0$$

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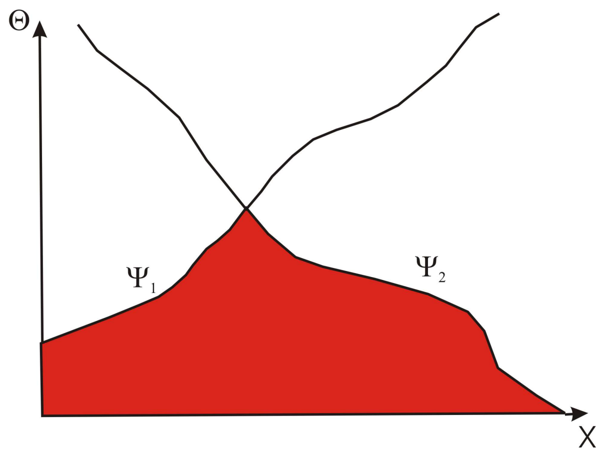

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Explicit construction of optimal policy based on Hamilton-Jacobi-Bellman equation:

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Question:

What if we wish to impose a dynamic upper bound $(\bar{\vartheta}_t)_{t\geq 0}$ on feasible investments? $\rightsquigarrow$ 'finite fuel' constraint, additional state variable(s)

General framework

Consider utility functional

$$U(\theta) = \mathbb{E}\int_0^\infty u(\theta_t)\,\mu(dt) - \mathbb{E}\int_0^\infty k_t\,d\theta_t$$

where

- ▷ μ atomless optional random measure on $[0, +\infty)$ with full support
- ▷ $u = u(\omega, t, \vartheta)$ random utility function: strictly concave in ϑ, $\mathbb{P}\otimes\mu$-integrable w.r.t. (ω, t)
- ▷ k optional process of class (D), continuous in expectation with $k_{+\infty} = 0$

Dynamic upper bound on admissible controls:

$$\theta \in \mathscr{A} \Leftrightarrow \theta_t \in [\underline{\vartheta}, \overline{\vartheta}_t] \text{ adapted, left-continuous, increasing}$$

General problem:

Maximize $U(\theta)$ subject to $\theta \in \mathscr{A}$.

First order conditions

Theorem. *Under some integrability conditions on u and k, a policy $\theta^* \in \mathscr{A}$ solves our irreversible investment problem iff*

(i) θ^* *is flat off* $\{\nabla U(\theta^*) = \mathbb{S}(\theta^*)\}$

(ii) $A(\theta^*)$ *is flat off* $\{\theta^* = \overline{\vartheta}\}$

where

- $\nabla U(\theta^*)_S = \mathbb{E}\left[\int_S^\infty u'(\theta^*_t)\,\mu(dt)\,\middle|\,\mathscr{F}_S\right] - k_S$ *('utility gradient')*
- $\mathbb{S}(\theta^*)_S = M(\theta^*)_S - A(\theta^*)_S$ *is the Snell envelope of* $\nabla U(\theta^*)$*:*

$$\mathbb{S}(\theta^*)_S = \operatorname*{ess\,sup}_{T \geq S} \mathbb{E}\left[\nabla U(\theta^*)_T \,\middle|\, \mathscr{F}_S\right]$$

Interpretation:

(i) $\leftrightsquigarrow$ invest only when 'maximal' marginal impact is guaranteed

(ii) $\leftrightsquigarrow$ be fully invested whenever marginal impact tends to decline

First order conditions

Proof.

By concavity:

θ^* optimal in $\mathscr{A} \Leftrightarrow \theta^* \in \arg\max_{\theta \in \mathscr{A}} \mathbb{E} \int_0^\infty \phi_t \, d\theta_t$ where $\phi = \nabla U(\theta^*)$

Lemma. *Let $\phi \geq 0$ be an optional process of class (D), continuous in expectation with $\phi_\infty = 0$, and let $\psi = M - A$ denote its Snell envelope. Then*

$$\theta^* \in \arg\max_{\theta \in \mathscr{A}} \mathbb{E} \int_0^\infty \phi_t \, d\theta_t \quad \textit{iff} \quad \begin{cases} \textit{(i)} & \theta^* \textit{ is flat off } \{\phi = \psi\} \\ \textit{(ii)} & A \textit{ is flat off } \{\theta^* = \overline{\vartheta}\} \end{cases}$$

Proof of Lemma.

$$\mathbb{E} \int_0^\infty \phi_t \, d\theta_t \leq \mathbb{E} \int_0^\infty \psi_t \, d\theta_t = \mathbb{E} \int_0^\infty (A_\infty - A_t) \, d\theta_t$$
$$= \mathbb{E} \int_0^\infty (\theta_t - \underline{\vartheta}) \, dA_t \leq \mathbb{E} \int_0^\infty (\overline{\vartheta}_t - \underline{\vartheta}) \, dA_t$$

□

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Question:

How to construct a policy θ^* satisfying these first order conditions?

Construction of the optimal policy

Theorem. *Let κ be an optional process such that*

$$k_S = \mathbb{E}\left[\int_S^\infty u'(\sup_{s\in[S,t)} \kappa_s)\,\mu(dt)\,\middle|\,\mathscr{F}_S\right] \quad (S \in \mathscr{T})\,.$$

Then

$$\theta^*_t = \sup_{s\in[0,t)} \{\kappa_s \wedge \overline{\vartheta}_s\} \vee \underline{\vartheta} \quad (t \geq 0)$$

maximizes

$$U(\theta) = \mathbb{E}\int_0^\infty u(\theta_t)\,\mu(dt) - \mathbb{E}\int_0^\infty k_t\,d\theta_t$$

over $\theta \in \mathscr{A}$.

Remark:

- Existence of κ as above can be guaranteed under additional technical assumptions on u'; see B.-El Karoui (2004)
- κ is universal in that it does not depend on $\underline{\vartheta}$ or $\overline{\vartheta}$

Construction of the optimal policy

Proof. Verify first order conditions for policy $\theta^*_t = \sup_{s\in[0,t)}\{\kappa_s \wedge \overline{\vartheta}_s\} \vee \underline{\vartheta}$!

$$\begin{aligned}
\mathbb{E}\left[\nabla U(\theta^*)_T \,|\, \mathscr{F}_S\right] &\le \mathbb{E}\left[\int_T^\infty \{u'(\sup_{s\in[0,t)}\{\kappa_s \wedge \overline{\vartheta}_s\}) - u'(\sup_{s\in[T,t)} \kappa_s)\}\,\mu(dt)\,\middle|\, \mathscr{F}_S\right] \\
&\le \mathbb{E}\left[\int_T^\infty \{u'(\sup_{s\in[0,t)}\{\kappa_s \wedge \overline{\vartheta}_s\}) - u'(\sup_{s\in[S,t)} \kappa_s)\}\,\mu(dt)\,\middle|\, \mathscr{F}_S\right] \\
&\le \mathbb{E}\left[\int_S^\infty \{u'(\sup_{s\in[0,t)}\{\kappa_s \wedge \overline{\vartheta}_s\}) - u'(\sup_{s\in[S,t)} \kappa_s)\} \vee 0\,\mu(dt)\,\middle|\, \mathscr{F}_S\right] \\
&= \mathbb{S}(\theta^*)_S \;\; \text{as '$=$' everywhere for } T = T_S = \inf\{t \ge S \mid \kappa_t > \overline{\vartheta}_{t+}\}
\end{aligned}$$

Flat off conditions:

$$dA(\theta^*)_S > 0 \Rightarrow T_S = S \text{ only optimal stopping time} \Rightarrow \theta^*_{S+} = \overline{\vartheta}_{S+}$$

$$d\theta^*_S > 0 \Rightarrow \sup_{[S,t)}\{\kappa \wedge \overline{\vartheta}\} = \sup_{[0,t)}\{\kappa \wedge \overline{\vartheta}\} > \underline{\vartheta} \text{ for } t > S \Rightarrow \nabla U(\theta^*)_S = \mathbb{S}(\theta^*)_S$$

□

Construction of the optimal policy

...via solving representation problem: ***Quadratic Monotone Follower***

Quadratic cost functional:

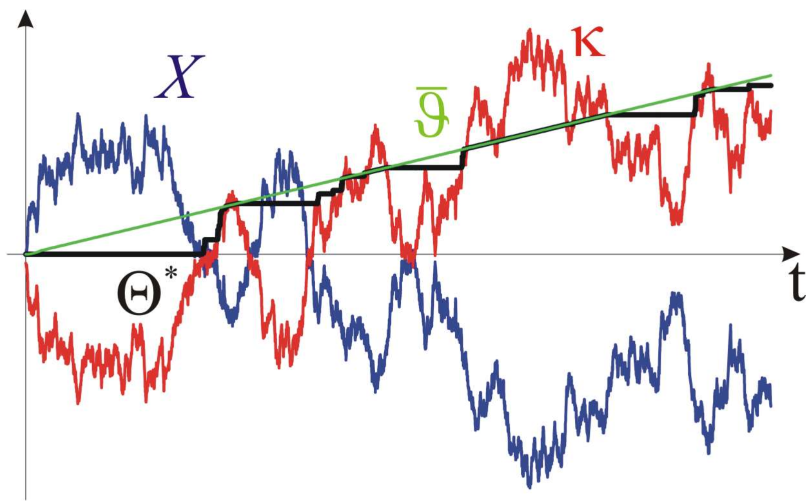
$$U(\theta) = -\mathbb{E}\int_0^\infty e^{-\delta t}\frac{1}{2}(X_t+\theta_t)^2\,dt$$

For X a Lévy-process, corresponding representation problem solved by

$$\kappa_s = c - X_s \quad \text{where} \quad c = \mathbb{E}\int_0^\infty \delta e^{-\delta t} \inf_{s\in[0,t)} X_s\,dt < 0$$

Hence:

$$\theta^*_s = \sup_{s\in[0,t)} \{(c - X_s)\wedge \overline{\vartheta}_s\} \vee \underline{\vartheta}$$: 'reflect controlled system at level c'



Construction of the optimal policy

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Hence:

$$\theta_s^* = \sup_{s \in [0,t)} \{(c - X_s) \wedge \overline{\vartheta}_s\} \vee \underline{\vartheta} \;:\; \text{'reflect controlled system at level } c\text{'}$$

Note:

- method works for large class of Lévy processes: no integro-variational inequalities needed
- arbitrary dynamic fuel constraint: compare Chow et al. (1985), Karatzas (1985), Karatzas and Shreve (1984,1985)

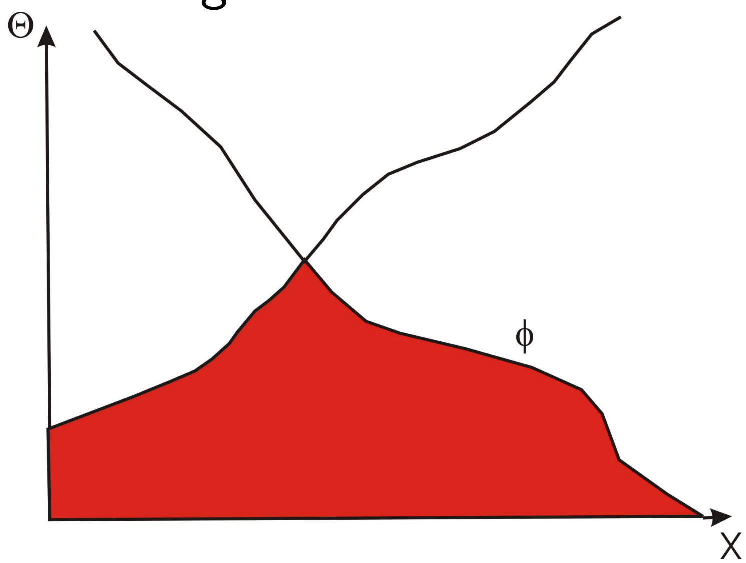
Construction of the optimal policy

...via solutions in unconstrained case: ***Irreversible investments***

Utility functional as in Kobila (1993):

$$U(\theta) = \mathbb{E} \int_0^\infty e^{-\delta t} u(X_t, \theta_t)\, dt$$

for X a geometric Brownian motion.



$$\kappa_s = \phi(X_s)$$

where

$$\phi(x) = \theta \quad \text{for} \quad x = \psi_i(\theta)$$

Note:

- arbitrary dynamic fuel constraint: DP without additional dimension for fuel
- Scheinkman-Zariphopoulou (2001) $\leftrightarrow$ Kobila (1993)

Open problems

- ▷ What if underlying stochastic dynamics depend on control policy?
- ▷ What if we have to respect a minimum investment level, i.e., an additional dynamic lower bound $(\underline{\vartheta}_t)_{t\geq 0}$?
- ▷ What about multi-dimensional controls?
- ▷
- ▷
- ▷

Costly disinvestment

Investments with transaction costs:

- Utility functional:
$$U(\theta) = \mathbb{E}\left[\int_0^\infty u(\theta_t)\,\mu(dt) - \int_0^\infty k_t\,d\theta_t^+ + \int_0^\infty l_t\,d\theta_t^-\right]$$
where l describes rebate for disinvestment.

- now: controls $\theta \in \mathscr{A}$ of *bounded variation*: $d\theta = d\theta^+ - d\theta^-$.

- No fuel constraint: $\overline{\vartheta} = +\infty$

Assumption. $k > l$ optional, of class (D), cont. in expect. with $k_\infty = l_\infty = 0$.

Optimal (Dis-)Investment:

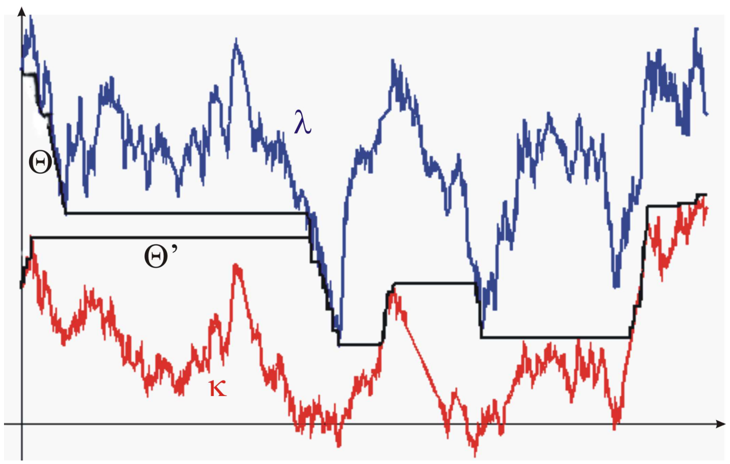
Maximize $U(\theta)$ subject to $\theta \in \mathscr{A}$.

Costly disinvestment

Theorem. *Let $\kappa < \lambda$ be optional processes such that*

$$\begin{cases} k_S = \mathbb{E}\left[\int_S^\infty u'(\Theta(S,\kappa_S;\kappa,\lambda)_t)\,\mu(dt)\,\middle|\,\mathscr{F}_S\right] \\ l_S \;= \mathbb{E}\left[\int_S^\infty u'(\Theta(S,\lambda_S;\kappa,\lambda)_t)\,\mu(dt)\,\middle|\,\mathscr{F}_S\right] \end{cases} \quad \text{for} \quad S \in \mathscr{T}$$

where $\Theta(S,x;\kappa,\lambda)$ denotes the bounded variation processes reflected at κ and λ starting in x at time S.



Then $\theta^* = \Theta(0,\underline{\vartheta};\kappa,\lambda)$ maximizes

$$U(\theta) = \mathbb{E}\left[\int_0^\infty u(\theta_t)\,\mu(dt) - \int_0^\infty k_t\,d\theta_t^+ + \int_0^\infty l_t\,d\theta_t^-\right]$$

over $\theta \in \mathscr{A}$.

Costly disinvestment

Characterizations of κ and λ:

- as solution to a coupled representation problem
- as universal stopping signals for family of Dynkin games:

$$\sup_S \inf_T \mathbb{E}\left[\int_0^{S\wedge T} u'(\vartheta)\,\mu(dt) - k_S 1_{\{S<T\}} + l_T 1_{\{T<S\}}\right] \quad (\vartheta \in \mathbb{R})$$

- as value of a non-standard Dynkin game:

$$\kappa_v = \operatorname*{ess\,sup}_{S>v} \operatorname*{ess\,inf}_{T>v} \Theta(v;S,T)$$

where $\Theta(v;S,T) \in \mathscr{F}_v$ solves

$$\mathbb{E}\left[\int_v^{S\wedge T} u'(\Theta)\,\mu(dt) \,\middle|\, \mathscr{F}_v\right] = \mathbb{E}\left[k_S 1_{\{S<T\}} - l_S 1_{\{T<S\}} \,\middle|\, \mathscr{F}_v\right]$$

Conclusion

- ▷ irreversible investment problem with a dynamic fuel constraint addressed by methods from convex analysis
- ▷ first order characterization of optimality in terms of Snell envelopes of optimal utility gradients
- ▷ construction of optimal policy in terms of solution to a representation problem
- ▷ universal control signal independent of constraint
- ▷ reduction of constrained to unconstrained problem
- ▷ problem with costly disinvestment: system of representation problems
- ▷ connection with Dynkin games and stopping signals