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Complete-market Models for Option Trading with Stochastic Volatility

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Agenda

- The Black-Scholes theory and its limitations
- Single-factor stochastic volatility models
- Multi-factor stochastic volatility models
- A direct approach
  - Markovian framework
  - A geometric condition for complete markets
- Implementation
- Concluding Remarks

1 The Black-Scholes theory

Based on a log-normal asset price driven by Brownian motion w_t

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}w_t$$

and a 'bank account' paying continuously compounded interest at rate r.

Classic case: the European call option $H = \max[S_T - K, 0]$. If $C(t, S_t)$ denotes the price at time $t \leq T$ then

- C is given by

  $$C(t, S) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

  where

  $$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

- C satisfies the PDE

  $$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \quad C(T, S) = [S - K]^+.$$

- The hedging strategy is given by

$$\phi_t = \frac{\partial C}{\partial S}(t, S_t) \quad \text{“Delta hedging”}.$$

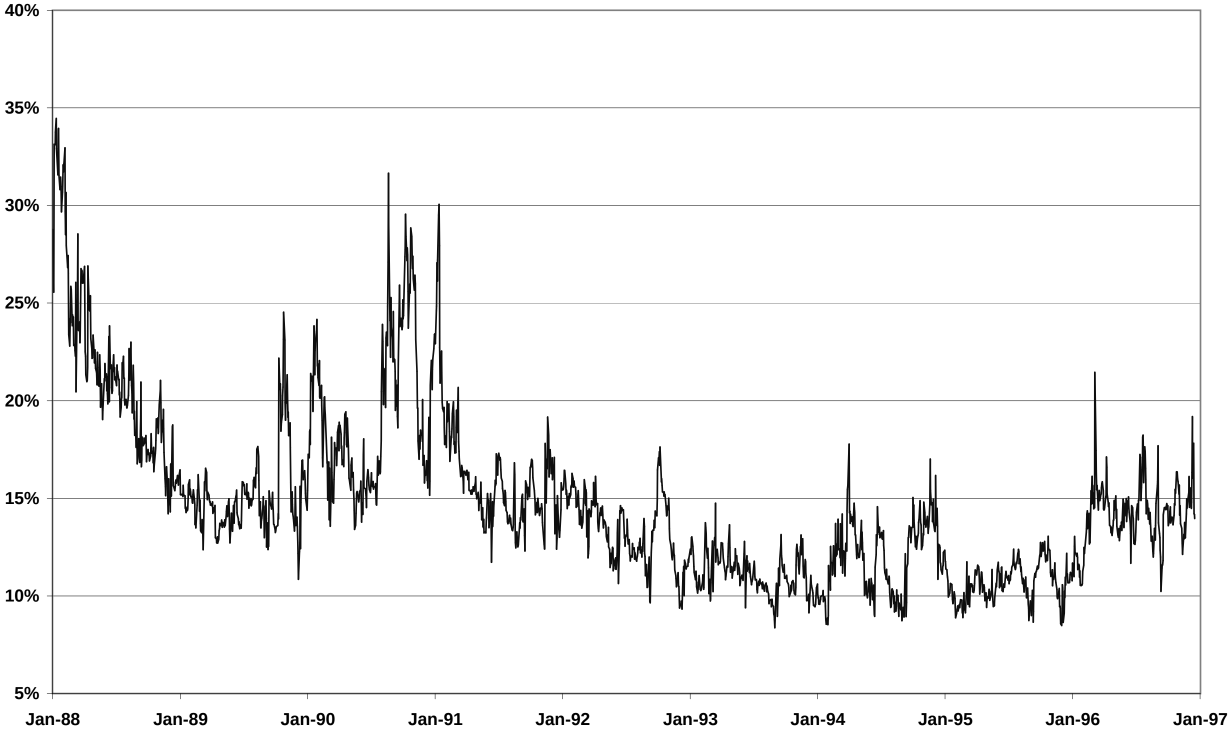
From Black-Scholes we have a 5-parameter formula $\mathrm{BS}(S, r, K, T, \sigma)$ for the option price. BS is a monotone increasing function of σ. In a listed options market the *price* p is observable. The *implied volatility* is $\hat{\sigma}$ satisfying

$$p = \mathrm{BS}(S, r, K, T, \hat{\sigma}).$$

Limitations

- $\hat{\sigma}$ not constant over different strikes and maturities.
- ATM volatility is “stochastic” (see chart).

THEME OF THE TALK: how do we adjust the model to retain the advantages of a ‘complete market’ while matching observed patterns of implied volatility better?



Implied volatility of ATM S&P500 index options, 1988–1997

2 First approach: Level-dependent Volatility

With a price model of the form

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(S_t)S_t\mathrm{d}w_t,$$

the market is complete and the unique price of a European option with exercise value H at time T is, as usual

$$E_Q\left[e^{-rT}H\right].$$

All options are redundant. Special cases: CEV model

$$dS_t = rS_t\mathrm{d}t + \beta(S_t)^{1-\alpha}\mathrm{d}w_t,$$

or 'implied tree' models (Derman and Kani, Dupire).

Choice of $\sigma(\cdots)$ gives price distributions that match observed volatility smiles. But no implications for hedging.

NOTE: volatility in single-factor models can 'look random'. Classic example: the 2-vector random variable

$$\begin{bmatrix} w_t \\ \int_0^t w_s ds \end{bmatrix}$$

has non-singular covariance matrix ($\rho = \sqrt{3}/2 = 0.866$)

$$\begin{bmatrix} t & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 \end{bmatrix}$$

Vega Hedging

The *vega* of an option C is $v = \partial C/\partial\sigma$, the sensitivity of the Black-Scholes value to changes in the volatility σ. If we hold option C (say an OTC option) we could in principle hedge the volatility risk by selling v/v' units of an exchange traded option C' whose vega is v', giving a 'vega neutral' portfolio $C-(v/v')C'$. Procedure is theoretically inconsistent in that the valuation method – Black-Scholes – assumes *no* variation in volatility.

1-factor models give *no information* about vega hedging.

3 Multi-factor Stochastic volatility models

Volatility parameter is treated as a stochastic process. Market model, in the physical measure P, is

$$\begin{aligned} \mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}w_t \\ \mathrm{d}\sigma(t) &= a(S(t),\sigma(t))\mathrm{d}t + b(S(t),\sigma(t))\mathrm{d}w_t^\sigma . \end{aligned}$$

$E\mathrm{d}w_t\mathrm{d}w_t^\sigma = \rho\mathrm{d}t$. Coefficients a, b define the volatility model. Examples: Hull and White (1987) and Heston (1993).

Let $w_t^\sigma = \rho w_t + \rho' w_t'$ where w_t' is a BM independent of w_t and $\rho' = \sqrt{1-\rho^2}$. Measures Q equivalent to P have densities

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(\int_0^T \Phi_s \mathrm{d}w_s - \frac{1}{2}\int_0^T \Phi_s^2 \mathrm{d}s + \int_0^T \Psi_s \mathrm{d}w_s' - \frac{1}{2}\int_0^T \Psi_s^2 \mathrm{d}s\right)$$

Choice of $\Phi = (r-\mu)/\sigma$ is mandatory, but no restrictions on $\Psi = \Psi(S,\sigma)$ since σ is not a traded asset.

Market is incomplete, with many martingale measures $Q(\Psi)$.

Under measure $Q(\Psi)$,

$$\begin{aligned} dS(t) &= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\widetilde{w}_t \\ d\sigma(t) &= \widetilde{a}(S(t),\sigma(t))\mathrm{d}t + b(S(t),\sigma(t))\mathrm{d}\widetilde{w}_t^\sigma \end{aligned}$$

where $\widetilde{w}$, $\widetilde{w}^\sigma$ are Q-Brownian motions with $E\mathrm{d}\widetilde{w}\mathrm{d}\widetilde{w}^\sigma = \rho\mathrm{d}t$ and $\widetilde{a}(S,\sigma) = a + b\rho\Phi + b\rho'\Psi$.

Take option written on $S(t)$ with exercise value $h(S(T))$ at time T. We *define* its value at $t < T$ to be

$$C(t,S(t),\sigma(t)) = E_Q\left[e^{-r(T-t)}h(S(T)) \middle| S(t),\sigma(t) \right].$$

Option value C then satisfies the PDE

$$\frac{\partial C}{\partial t} + rs\frac{\partial C}{\partial s} + \widetilde{a}\frac{\partial C}{\partial \sigma} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial \sigma^2} + \rho\sigma s b\frac{\partial^2 C}{\partial s \partial \sigma} - rC = 0$$

Process $O(t) := C(t, S(t), \sigma(t))$ satisfies

$$\begin{aligned} \mathrm{d}O(t) &= rO(t)\mathrm{d}t + \frac{\partial C}{\partial s}\sigma S \mathrm{d}\widetilde{w} + \frac{\partial C}{\partial \sigma} b \mathrm{d}\widetilde{w}^{\sigma}. \\ &= rO(t)\mathrm{d}t + g(t, S(t), \sigma(t))d\hat{w}(t). \end{aligned}$$

If the map $\sigma \mapsto o = C(t, s, \sigma)$ is invertible, so that $\sigma = D(t, s, o)$, then

$$g(t, s, \sigma) = g(t, s, D(t, s, o) \equiv F(t, s, o)o,$$

giving price equations

$$\begin{aligned} dS_t &= rS_t\mathrm{d}t + D(t, S_t, O_t)S_t\mathrm{d}w_t \\ dO_t &= rO_t\mathrm{d}t + F(t, S_t, O_t)O_t\mathrm{d}\hat{w}_t \end{aligned}$$

S_t and O_t are linked by boundary condition $O(T) = h(S(T))$. Creates a complete market model with traded assets S_t, O_t for which Q is the unique EMM. Note that implied volatility $\widehat{\sigma}_t$ satisfies

$$O_t = \mathrm{BS}(S_t, K, r, \widehat{\sigma}_t, T - t).$$

Model non-unique, choice of Ψ only affects option vol F.

4 A Direct Approach

Suppose market contains n exchange-traded European call options on S_t with maturity times T_i and strikes $K_i, i = 1, \dots, n{-}1$, with price processes $O_1(t), \dots, O_{n-1}(t)$. Probability space is $(\Omega, \mathcal{F}, \mathcal{F}_t, w_t, P)$ (w_t is BM in R^n). Here P is the *risk-neutral* measure.

Let $Y > 0$ be an integrable $\mathcal{F}_{T_{n-1}}$-measurable random variable and *define*

$$
\begin{aligned}
S(t) &= E[e^{-r(T_{n-1}-t)} Y | \mathcal{F}_t], \quad t \in [0, T_{n-1}] \\
O_i(t) &= E[e^{-r(T_i - t)} [S(T_i) - K_i]^+ | \mathcal{F}_t], \quad t \in [0, T_i], \; i = 1, \dots, n-1
\end{aligned}
$$

In particular

$$O_{n-1}(t) = E[e^{-r(T_{n-1}-t)} [Y - K_i]^+ | \mathcal{F}_t].$$

Model automatically specifies a model for implied volatilities $\widehat{\sigma}_i$ of the options, which satisfy

$$O_i(t) = \mathrm{BS}(S(t), K_i, r, \widehat{\sigma}_i(t), T_i - t) \quad i = 1, \ldots, n-1. \tag{1}$$

Alternative approach (Schönbucher): model $S, \widehat{\sigma}_i$ and *define* $O_i(t)$ by (1).

- SDEs for $\widehat{\sigma}_i$ must satisfy a drift condition to exclude arbitrage.

- Quite awkward behaviour close to maturity. (NB: using BS is *not* entirely arbitrary!)

5 A Markovian framework

Suppose $\xi_t \in R^n$ is the unique solution of the non-degenerate SDE

$$\mathrm{d}\xi_t = m(\xi_t)\mathrm{d}t + \Sigma(\xi_t)\mathrm{d}w_t.$$

Associated with this is the backward equation, for a function $v(t,x)$:

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{A}v - rv &= 0, \quad (t,x) \in [0,T] \times R^n \\ v(T,x) &= h(x). \end{aligned}$$

Here h is given boundary data at some terminal time T and $\mathcal{A}$ is the differential generator of ξ_t:

$$\mathcal{A}f(x) = \nabla f(x)\, m(x) + \frac{1}{2}\sum_{i,j}(\Sigma(x)\Sigma^{\mathrm{T}}(x))_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

By Feynman-Kac the solution of the PDE is

$$\begin{aligned} v(t,x) &= P_{T-t}h(x) \\ &= E_{t,x}\left[e^{-r(T-t)}h(\xi_T)\right]. \end{aligned}$$

We take ξ_t as the process of 'factors' underlying the financial market and suppose that Y takes the form $Y = h_0(\xi_{T_{n-1}})$. Price process is now

$$S_t = P_{T_{n-1}-t}h_0(\xi_t) =: v_0(t,\xi_t).$$

Option values are given for $t \le T_1$ by

$$O_i(t) = P_{T_i-t}h_i(\xi_t) =: v_i(t,x_t), \quad i = 1, , \ldots, n-1,$$

where

$$h_i(x) = [P_{T_{n-1}-T_i}h_0(x) - K_i]^+.$$

Discounted asset price satisfies

$$\mathrm{d}\left(e^{-rt}S_t\right) = e^{-rt}\nabla v_0 \Sigma \mathrm{d}w,$$

and

$$\mathrm{d}\left(e^{-rt}O_i(t)\right) = e^{-rt}\nabla v_i \Sigma \mathrm{d}w, \quad i = 1, \ldots, n-1.$$

(Note: ∇v always denotes a row vector.)

We want to show that this market is complete, i.e. any other contingent claim maturing at time $T \leq T_1$ can be hedged using a portfolio of cash, underlying S and options $O_1, \ldots, O_{n-1}$.

A *self-financing portfolio* with value X_t at time t is constructed in the following way from the trading stategy α: the increment of portfolio value is

$$\mathrm{d}X_t = \alpha_0(t)\mathrm{d}S_t + \sum_{i=1}^{n-1} \alpha_i(t)\mathrm{d}O_i(t) + \left(X_t - \alpha_0(t)S(t) - \sum_1^{n-1} \alpha_i(t)O_i(t) \right) r\mathrm{d}t.$$

In discounted units $\hat{X}_t = e^{-rt}X_t$,

$$\begin{aligned} \mathrm{d}\hat{X}_t &= \alpha_0(t)\mathrm{d}\hat{S}_t + \sum_1^{n-1} \alpha_i(t)\mathrm{d}\hat{O}_i(t) \\ &= e^{-rt} \left(\sum_{i=0}^{n-1} \alpha_i(t)\nabla v_i \right) \Sigma \mathrm{d}w. \end{aligned}$$

Proposition 1 *Suppose that the matrix*

$$G(t,x) = \begin{bmatrix} \nabla v_0(t,x) \\ \nabla v_1(t,x) \\ \vdots \\ \nabla v_{n-1}(t,x) \end{bmatrix}$$

is nonsingular for all $(t,x) \in [0,T] \times R^n$. *Then we have a complete market model.*

Proof follows from the martingale representation theorem for $w(t)$. The trading strategy to replicate a general contingent claim H is

$$\alpha_t = e^{rt} \chi_t \Sigma^{-1}(\xi_t) G^{-1}(t, \xi_t). \tag{2}$$

where χ_t comes from the martingale representation of H.

Special case: suppose that $H = h(\xi_T)$ and define

$$v(t,x) = P_{T-t}h(x), \quad t \leq T.$$

Then the replicating strategy is given by

$$\alpha(t) = \nabla v(t,\xi_t)G^{-1}(t,\xi_t).$$

Implementation: To implement these trading strategies, we need to know the value of the factor process ξ_t, but this is not directly observed: the market data consists of the traded asset prices $(S_t, O_1(t), \ldots, O_n(t))$ and we must recover the state vector ξ_t from these.

6 The non-singularity condition

Consider an n-dimensional case where the factor process is a non-degenerate diffusion $(\xi_t, 0 \le t \le T)$ in R^n, the exercise value functions $h_i(x)$ are such that $E|h_i(\xi_T)| < \infty,\ i = 1, \ldots, n$, and the exercise time is T for each option. Define functions v_i by

$$v_i(t,x) = E_{t,x}[h_i(\xi_T)],$$

and let $G(t,x)$ be the $n \times n$ matrix

$$G(t,x) = \begin{bmatrix} \nabla v_1(t,x) \\ \vdots \\ \nabla v_n(t,x) \end{bmatrix}, \tag{3}$$

where ∇ denotes the gradient in the x variables. Consider first the Brownian case.

Proposition 2 *Suppose ξ_t is Brownian motion in R^n. Then $G(t,x)$ defined by (3) is non-singular if and only if there exists no non-zero vector $\alpha \in R^n$ such that*

$$H_\alpha \perp \mathcal{L}\{\xi_T^1, \ldots, \xi_T^n\}.$$

Here $\mathcal{L}\{\cdots\}$ denotes the linear subspace spanned by the indicated random variables in $L_2(\Omega, \mathcal{F}_T, P_{t,x})$, and

$$H_\alpha = \sum_{k=1}^{n} \alpha_k h_k(\xi_T).$$

Proof: For $0 \leq \tau \leq T$ we have

$$v_i(T-\tau, x) = c \int_{-\infty}^{\infty} h_i(y) e^{-|y-x|^2/2\tau} \mathrm{d}y,$$

where $c = 1/(2\pi\tau)^{n/2}$.

We have

$$\begin{aligned}\frac{\partial v_i}{\partial x_j}(t,x) &= \frac{c}{\tau}\int_{-\infty}^{\infty} h_i(y)(y_j - x_j)e^{-|y-x|^2/2\tau}\mathrm{d}y \\ &= \frac{1}{\tau}E_{t,x}[h_i(\xi_T)(\xi_T^j - x_j)] \\ &= \frac{1}{\tau}\mathrm{cov}_{t,x}(h_i(\xi_T), \xi_T^j).\end{aligned}$$

Thus

$$\nabla v_i(t,x) = \frac{1}{\tau}\mathrm{cov}(H_i, \xi_T) \tag{4}$$

where $H_i = h_i(\xi_T)$ and $\mathrm{cov}(\cdots)$ denotes componentwise covariance. Defining H_α as above we therefore have

$$\tau G\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{cov}(H_i, \xi_T) = \mathrm{cov}(H_\alpha, \xi_T),$$

so that G is singular when $H_\alpha = 0$ or H_α is, for some $\alpha \neq 0$, orthogonal to the linear span $\mathcal{L}\{\xi_T^1, \ldots, \xi_T^n\}$.

The proposition shows that G is in some sense 'generically' non-singular. Special cases:

(i) G is singular at all (t, x) if either the functions h_i are linearly dependent (so $H_\alpha \equiv 0$ for some α).

(ii) G is singular if h_i functions do not depend on all coordinates of x. For example, if $h_i(x) = \tilde{h}_i(x_2, \dots, x_n)$, $i = 1, \dots, n$ for some functions $\tilde{h}_i$, then $\mathrm{cov}(H_i, \xi_T^1) = 0$, so that $\mathrm{rank}(G) \leq n - 1$.

(iii) Let $h_i(x) = x_1^2 + 2cx_1$ for some constant c. Then

$$E_{t,x}[h_1(\xi_T)(\xi_T^1 - x_1)] = 2(T - t)(x_1 + c),$$

while plainly

$$E_{t,x}[h_1(\xi_T)(\xi_T^k - x_k)] = 0, \quad k > 1.$$

Thus G is singular on the subspace $\{x : x_1 + c = 0\}$.

7 Non-degenerate diffusion processes

Generalization uses Bismut formula, cf Bismut 1984, Elworthy & Li 1994, Elliott & Kohlmann 1989.

Suppose the process $\xi_{s,t}$ satisfies the SDE

$$\mathrm{d}\xi_{s,t}(x) = m(\xi_{s,t}(x))\mathrm{d}t + \sum_{i=1}^{n} \sigma_i(\xi_{s,t}(x))\mathrm{d}w_t^i, \quad \xi_{s,s} = x$$

where $w_t = (w_t^1, \dots, w_t^n)$ is Brownian motion in R^n and m, σ_i are smooth functions with bounded derivatives. Then the map $x \mapsto \xi_{s,t}(x)$ is smooth, and the Jacobian $D_{s,t} = \partial\xi_{s,t}(x)/\partial x$ satisfies

$$\mathrm{d}D_{s,t} = \frac{\partial m}{\partial x} D_{s,t}\,\mathrm{d}t + \sum_{1}^{n} \frac{\partial \sigma_i}{\partial x} D_{s,t}\,\mathrm{d}w_t^i, \quad D_{s,s} = I.$$

The Clark-Haussmann-Ocone formula gives the following stochastic integral representation of a random variable $h(\xi_{s,T}(x))$:

$$\xi_{s,T}(x)) = E_{s,x}[h] + \int_s^T E[\nabla h(\xi_{s,T}(x))D_{s,T}|\mathcal{F}_t]D_{s,t}^{-1}\Sigma(\xi_{s,t}(x))\mathrm{d}w_t$$

(Σ is the matrix whose i'th column is σ_i.) This gives the 'integration by parts' formula: for a vector process u such that $E_{s,x}\int_s^T |u_t|^2\mathrm{d}t < \infty$ we have

$$\sum_1^n E\left[h(\xi_{s,T}(x))\int_s^T u_t^i \mathrm{d}w_t^i\right] = \int_s^T E\left[\nabla h(\xi_{s,T}(x))D_{s,T}D_{s,t}^{-1}\Sigma(\xi_{s,t}(x))u_t\mathrm{d}t\right].$$

If U_t is an $n \times n$ matrix-valued process each of whose components satisfies the integrability condition then

$$E\left[h(\xi_{s,T}(x))\int_s^T \mathrm{d}w_t^{\mathrm{T}} U_t\right] = \int_s^T E\left[\nabla h(\xi_{s,T}(x)) D_{s,T} D_{s,t}^{-1} \Sigma(\xi_{s,t}(x)) U_t \mathrm{d}t\right].$$

Define $v(s,x) = E[h(\xi_{s,T}(x))]$; then $\nabla v(s,x) = E[\nabla h(\xi_{s,T}(x)) D_{s,T}]$, and taking $U_t = \Sigma^{-1} D_{s,t}$, so that $D_{s,t}^{-1}\Sigma U = I$. We obtain the *Bismut formula*:

$$\nabla v(s,x) = \frac{1}{T-s} E\left[h(\xi_{s,T}(x))\int_s^T \mathrm{d}w_t^{\mathrm{T}} \Sigma^{-1}(\xi_{s,t}(x)) D_{s,t}\right]$$

The formula is easily extended to non-smooth h by continuity.

Proposition 3 *For fixed (s,x) define random variables $Y_1,\dots,Y_n$ by*

$$Y_j = \sum_{i=1}^{n} \int_s^T \left(\Sigma^{-1}(\xi_{s,t}(x))D_{s,t}\right)_{ij} \mathrm{d}w_t^i.$$

Then $G(s,x)$ is non-singular if and only if there exists no non-zero vector $\alpha \in R^n$ such that

$$H_\alpha \perp \mathcal{L}\{Y_1,\dots,Y_n\} \tag{5}$$

in $L_2(\Omega, \mathcal{F}_T, P_{s,x})$, where

$$H_\alpha = \sum_{k=1}^{n} \alpha_k h_k(\xi_{s,T}(x)).$$

PROOF: For $\alpha \in R^n$

$$
\begin{aligned}
(T-s)\alpha G(s,x) &= \sum_{i=1}^{n} \alpha_i E\left[h(\xi_{s,T}(x)) \int_s^T \mathrm{d}w_t^{\mathrm{T}} \Sigma^{-1}(\xi_{s,t}(x)) D_{s,t}\right] \\
&= (\mathrm{cov}(H_\alpha, Y_1), \ldots, \mathrm{cov}(H_\alpha, Y_1)).
\end{aligned}
$$

The result follows. $\diamondsuit$

Elworthy & Li (1994) give the Bismut formula in a geometric setting. Singularity of G can be interpreted in terms of vanishing of the first component of a certain Wiener chaos expansion of H_α.

8 Implementation

Set $r = 0$ and denote $\eta_1(t) = S(t), \eta_i(t) = O_{i-1}(t), i = 2, \ldots, n$. Then η_t is the vector of traded asset prices. It is a martingale and

$$\eta_t = v(t, \xi_t)$$

where the factor process ξ_t satisfies

$$\mathrm{d}\xi_t = m(\xi_t)\mathrm{d}t + \Sigma(\xi_t)\mathrm{d}w_t.$$

The hedge parameters α_t for an option with exercise value $h(\xi_T)$ are given by

$$\alpha_t = \nabla u(t, \xi_t) G^{-1}(t, \xi_t), \tag{6}$$

where $u(t, x) = P_{T-t}h(x)$.

Problem: (6) is expressed in the *wrong coordinates* (ξ not η).

8.1 Change of coordinates

Write ξ_t in coordinate-free form as

$$\mathrm{d}f(\xi_t) = \mathcal{X}_0 f(\xi_t)\mathrm{d}t + \sum_k \mathcal{X}_k f(\xi_t) \circ \mathrm{d}w_t^k,$$

where

$$\mathcal{X}_0 f(x) = \sum_i m_i(x) \frac{\partial f}{\partial x_i}$$

etc. Then $\eta_t = v(t, \xi_t)$ satisfies

$$\mathrm{d}f(\eta_t) = \sum_k v_* \mathcal{X}_k f(\eta_t) \mathrm{d}w_t^k,$$

where $v_* \mathcal{X}_k f(y) = \mathcal{X}_k (f \circ v)(t, v^{-1}(t, y))$.

- The good news: we now have an SDE directly for the traded assets.
- The bad news: we still have to compute $v^{-1}(t, y)$ at all points along the trajectory.

8.2 Tracking the state variable

We have $\mathrm{d}\eta = G\Sigma \mathrm{d}w$ and hence

$$\mathrm{d}\xi_t = m(\xi_t)\mathrm{d}t + \Sigma(\xi_t)\mathrm{d}w_t = m(\xi_t)\mathrm{d}t + G^{-1}(t, \xi_t)\mathrm{d}\eta_t,$$

an SDE driven by observed prices η_t. Proceed as follows:

1. Compute $\hat{x}_0 = v^{-1}(0, y_0)$, where y_0 is vector of time-0 prices.
2. Solve SDE
$$\mathrm{d}\hat{\xi}_t = m(\hat{\xi}_t)\mathrm{d}t + G^{-1}(t, \hat{\xi}_t)\mathrm{d}\eta_t, \quad \hat{\xi}_0 = \hat{x}_0. \tag{7}$$
3. Form hedge portfolio with parameters $\alpha = \nabla u(t, \hat{\xi}_t)G^{-1}(t, \hat{\xi}_t)$.

If $\hat{\xi}_t \equiv \xi_t$ then we have a 'perfect hedge' (within the model). There are stability questions when $\hat{x_0} \neq x_0$ and (7) is solved numerically.

Approaches

- Stochastic numerical analysis à la Kloeden-Platen.

- Nonlinear observers

Linear system case

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Hx(t) \end{aligned}$$

Observer is a dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - H\hat{x}(t)).$$

Error $e = \hat{x} - x$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}e(t) = (A - KH)e(t).$$

System is *stabilizable* if there exists K such that $A - KH$ is stable. Then $e(t) \to 0$ exponentially fast.

9 Concluding Remarks

- We have provided a theory of 'vega hedging': *all* traded assets are (in principle) used in the hedge portfolio.
- Many practical questions of numerical stability etc.
- Big modelling question: how to choose the factor process. The main drawback of the approach is that the modelling of 'volatility of volatility' is too indirect, leaving an awkward calibration problem.