Symmetric Chaos in a Local Codimension Two Bifurcation

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The group action

The action of $D_4$ on $\mathbb{R}^3$ is defined by

$$\rho : (x, y, w) \mapsto (-y, x, w), \quad \kappa : (x, y, w) \mapsto (x, -y, -w).$$

The isotropy subgroups are given up to conjugacy by $D_4$, $Z_4$, $D_1^e$, $D_1^v$ and 1, with the following lattice of inclusions:

$Z_4 = \langle \rho \rangle$ \quad $D_1^e = \langle \kappa \rangle$ \quad $D_1^v = \langle \kappa \rho \rangle$

Fix $Z_4 = \{(0, 0, w)\}$ \quad Fix $D_1^e = \{(x, 0, 0)\}$ \quad Fix $D_1^v = \{(x, x, 0)\}$
The vector field

**Proposition 0.1** The general smooth $D_4$-equivariant mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has the form

\[
\begin{aligned}
f_1(x, y, w) &= h_1(x^2, y^2, w^2)x - h_2(x^2, y^2, w^2)wy \\
f_2(x, y, w) &= h_1(y^2, x^2, w^2)y + h_2(y^2, x^2, w^2)wx \\
f_3(x, y, w) &= h_3(x^2 + y^2, x^2y^2, w^2)w + h_4(x^2 + y^2, x^2y^2, w^2)xy(x^2 - y^2)
\end{aligned}
\]

where $h_1, h_2, h_3, h_4$ are smooth real-valued functions.

We consider the truncated vector field

\[
\begin{aligned}
\dot{x} &= (\lambda - x^2 + by^2 + dw^2)x - wy \\
\dot{y} &= (\lambda - y^2 + bx^2 + dw^2)y + wx \\
\dot{w} &= (\mu + c(x^2 + y^2) - w^2)w + exy(x^2 - y^2)
\end{aligned}
\]

\[\text{(0.1)}\]

In the usual way, certain (generically nonzero) coefficients can be normalised to $\pm 1$ and we have chosen $-1$ to ensure that certain primary branches are supercritical.
SUMMARY OF THIS TALK

- Problem presentation;
- Primary and secondary branches (standard analytic methods);
- Tertiary branches (numeric methods: AUTO, DsTool, Lyapunov exponents, symmetry detectives);
- Summary of results.
Primary and secondary bifurcations

Primary bifurcations

The following table lists the three primary branches of equilibria, showing the branching equations and the eigenvalues for the equilibria.

<table>
<thead>
<tr>
<th>Isotr. $\Sigma$</th>
<th>Fix $\Sigma$</th>
<th>Branching equation</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>$(0, 0, w)$</td>
<td>$\mu = w^2$</td>
<td>$-2\mu$, $\lambda + d\mu \pm i\sqrt{\mu}$</td>
</tr>
</tbody>
</table>
| $\mathcal{D}_1$ | $(x, 0, 0)$  | $\lambda = x^2$    | $-2\lambda$, eigenvalues of $2 \times 2$ matrix with
|                |              |                    | $\text{tr} = \mu + (1 + b + c)\lambda$
|                |              |                    | $\text{det} = \lambda \{(1 + b)\mu + ((1 + b)(c - e))\lambda\}$ |
| $\mathcal{D}_1$ | $(x, x, 0)$  | $\lambda = (1 - b)x^2$ | $-2\lambda$, eigenvalues of $2 \times 2$ matrix with
|                |              |                    | $\text{tr} = \mu - 2(1 + b - c)x^2$
|                |              |                    | $\text{det} = -2x^2((1 + b)\mu + 2((1 + b)(c - e))x^2)$ |

Table 1: Branching equations and stability assignments for the primary branches of equilibria.
Proposition 0.2 The $\mathbb{Z}_4$ equilibria exist for $\mu > 0$. They are asymptotically stable if $\lambda + d\mu < 0$ and unstable if $\lambda + d\mu > 0$.

The $\mathbb{D}_1^e$ equilibria exist for $\lambda > 0$. They are asymptotically stable if

$$\mu + (1 + b + c)\lambda < 0, \quad (1 + b)\mu + ((1 + b)c - e)\lambda > 0,$$

and are unstable if one or both of these inequalities is reversed.

The $\mathbb{D}_1^s$ equilibria exist for $\lambda > 0$ provided $b < 1$, in which case they are asymptotically stable if

$$(1 - b)\mu - 2(1 + b - c)\lambda < 0, \quad (b^2 - 1)\mu + 2(e - (1 + b)c)\lambda > 0,$$

and are unstable if one or both of these inequalities is reversed.
Secondary bifurcations

**Proposition 0.3** (a) Secondary branches of equilibria bifurcate from the primary branches of equilibria as follows:

\[ Z_4 \quad \text{None.} \]

\[ D_1^1 \quad \text{At } (1 + b)\mu + ((1 + b)c - e)\lambda = 0. \]

\[ D_1^2 \quad \text{At } (b^2 - 1)\mu + 2(e - (1 + b)c)\lambda = 0. \]

These secondary bifurcations are pitchfork bifurcations and the bifurcating equilibria have trivial isotropy.

(b) Secondary branches of periodic solutions bifurcate from the primary branches of equilibria as follows:

\[ Z_4 \quad \text{At } \lambda + d\mu = 0. \]

\[ D_1^c \quad \text{At } \mu + (1 + b + c)\lambda = 0 \quad \text{provided } (1 + b)^2 + e < 0. \]

\[ D_1^v \quad \text{At } (1 - b)\mu - 2(1 + b - c)\lambda = 0 \quad \text{provided } -(1 + b)^2 + e > 0. \]

The resulting periodic solutions have trivial spatial symmetry and spatiotemporal symmetry \( Z_4, D_1^c \), and \( D_1^v \) respectively. For example, the \( Z_4 \) branch has quarter-period phase shift symmetry coupled with the action of \( \rho \).
Spatiotemporal $\mathbb{Z}_4$ symmetry:

$$(x, y, w)(t + T/4) = \rho.(x, y, w)(t) = (-y, x, w)(t)$$

Spatiotemporal $\mathbb{D}_1^c$ symmetry:

$$(x, y, w)(t + T/2) = k.(x, y, w)(t) = (x, -y, -w)(t)$$

**Lemma 0.4** If $b + 4cd < 3$, then the $\mathbb{Z}_4$ periodic solutions bifurcate supercritically (for $\lambda + d\mu > 0$) and are asymptotically stable. If $(1 + b)^2 + e < 0$, and $(1 + b)(2b^3 + 3b^2 - 3b + 2be + 3bc + 2ce + 2b^2c) > 0$, then the $\mathbb{D}_1^c$ periodic solutions bifurcate supercritically (for $\mu + (1 + b + c)\lambda > 0$) and are asymptotically stable.

If the appropriate inequality is reversed, then the corresponding periodic solutions exist subcritically and are unstable.
Tertiary bifurcations

From now on, we specify the values of the constants $b, c, d, e$ in the vector field (0.1) as follows:

\[ b = 0.9 \quad c = -2.1 \quad d = -0.05 \quad e = -19.2 \]

\[
\begin{align*}
\dot{x} &= (\lambda - x^2 + 0.9 y^2 - 0.05 w^2)x - wy \\
\dot{y} &= (\lambda - y^2 + 0.9 x^2 - 0.05 w^2)y + wx \\
\dot{w} &= (\mu - 2.1 (x^2 + y^2) - w^2)w - 19.2 xy(x^2 - y^2)
\end{align*}
\]

(0.2)

We concentrate on the positive quadrant $\lambda, \mu > 0$ of parameter space.
For those values of $b$, $c$, $d$ and $e$, there are:

<table>
<thead>
<tr>
<th>primary branches of equilibria</th>
<th>$\mathbb{Z}_4$ (stable) $\rightarrow$ $\mathbb{Z}_4$ periodic with Hopf b. at $\mu = 20\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathcal{D}_1^c$ (stable) $\rightarrow$ $\mathcal{D}_1^c$ periodic with Hopf b. at $\mu = 0.2\lambda$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{D}_1^v$ (unstable)</td>
</tr>
</tbody>
</table>
FIGURE 1. SCHEMATIC BIFURCATION DIAGRAM FOR SYSTEM (0.2).
$\mathbb{Z}_4$ periodic solutions

- these solutions can be computed numerically by constructing a modified Poincaré map $P$:
  - $X$ - local two-dimensional cross-section
  - $g : X \mapsto \rho X$ first hit map
  - $P = \rho^{-1} g : X \mapsto X$

- they correspond to fixed points of $P$;

- working with $P$ is numerically more efficient than working with the usual Poincaré map.
$\mathbb{Z}_4$ periodic solutions

We used AUTO to:

- find fixed points for the map $P$;

- compute a path of stable periodic solutions by increasing $\lambda$;

- detect the loss of stability of the solutions at a turning point;

- trace the path of turning points in the $(\lambda, \mu)$-space, using two-parameter continuation.
$D^c_1$ periodic solutions

\[ P_2 = k g_2 : X_2 \leftrightarrow X_2 \]

Using AUTO:

- we found fixed points for the map $P_2$;

- we found that the path of stable fixed points for $P_2$ loses stability via a period-doubling bifurcation;

- the pitchfork bifurcation to nonsymmetric periodic solutions is subcritical, resulting in nonstable solutions initially;

- almost immediately there is a turning point at which they regain stability;

- the corresponding hysteretic region of bistability is extremely thin;

- the turning point is quickly followed by a period-doubling cascade.
FIGURE 2. SECONDARY AND TERTIARY TRANSITIONS FOR THE VECTOR FIELD (0.2).
TERTIARY TRANSITIONS USING DsTool (fix $\lambda = 0.16$)

- Hopf bif.
- Subcritical pitchfork bif.
- Period-doubling bif.
- Nonsymmetric chaotic attractor
- Symmetry-increasing bif. to $D_4$
- $D_4^\circ$ symmetry, c.a. dominate
- Turning point
- $D_4^\circ$ eq.
- $D_4$ p.o.
- $\mu = 0.68$
- $\mu = 0.6835$
- $\mu = 0.6869$
- $\mu = 0.688$
- $\mu = 0.6891$
- $\mu = 0.72$
- $\mu = 0.739$
- $\mu = 0.73$
- $\mu = 0.74$
- $\mu = 0.8186$
- $\mu = 0.82$
- $Z_4$ eq.
- $Z_4$ p.o.

1. Complicated region with numerous transitions between periodic/chaotic solutions with/without $D_4^\circ$ symmetry.
Figure 6: Projection into the $(x, y)$- and $(x, w)$-planes of the $D_4$ symmetric periodic solution at $\lambda = 0.16$, $\mu = 0.68$. The plot includes 7,000 data points gathered with time step 0.01.

Figure 7: Projection into the $(x, y)$- and $(x, w)$-planes of the $Z_4$ symmetric periodic solution at $\lambda = 0.16$, $\mu = 0.82$. The plot includes 7,000 data points gathered with time step 0.01.
Figure 8: Projection into the $(x, y)$- and $(x, w)$-planes of the $D_4$ symmetric attractor at $\lambda = 0.16$, $\mu = 0.74$. The plot includes 200,000 data points gathered with time step 0.01.

Figure 9: Amalgamation of the plots of the the $Z_4$ symmetric and $D_4^1$ symmetric periodic solutions shown in Figures 6 and 7, together with their symmetric images.
Figure 10: Projection into the $(x,z)$- and $(x,w)$-planes of the symmetric attractor at $t = 1.0$. The plot includes 200,000 data points sampled with time step 0.01.

Figure 11: Projection into the $(x,y)$-plane of the symmetric and asymmetric periodic solutions at $t = 1.0$. Each plot includes 5,000 data points.
Figure 3: Region of $(\lambda, \mu)$ parameter space with positive Lyapunov exponent for the vector field \((2.1)\) with $b = 0.9$, $c = -2.1$, $d = -0.05$, $e = -19.2$. 
Figure 4: Symmetry types of attractors in $(\lambda, \mu)$ parameter space for the vector field (2.1) with $b = 0.9$, $c = -2.1$, $d = -0.05$, $e = -19.2$. Solid symbols denote chaotic, clear denote nonchaotic. (The $D_4$ symmetric attractors are all chaotic.)
Figure 5: Blow up of subregion of parameter space in Figure 4.
References


