Network topology and delay-induced oscillator death

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Coupled oscillators

\[ \begin{align*}
\dot{y}_1 + \varepsilon(y_1^2 - 1) \dot{y}_1 + y_1 &= \varepsilon \kappa (\dot{y}_2(t - \tau_0) - \dot{y}_1(t)) \\
\dot{y}_2 + \varepsilon(y_2^2 - 1) \dot{y}_2 + y_2 &= \varepsilon \kappa (\dot{y}_1(t - \tau_0) - \dot{y}_2(t))
\end{align*} \]

Synchronization ($\tau_0 = 0$)  
Death ($\tau_0 = 1$)

Death occurs only if $\tau_0 > 0$. 
Oscillator (amplitude) death

- Individual oscillatory units may stop oscillating when coupled
- Lord Rayleigh, Theory of Sound (19th century)
- Chemical oscillators (Bar-Eli 1985)
- Death by frequency mismatch (Mirollo, Strogatz 1990)
- Death not possible if identical systems are coupled (Aronson, Ermentrout, Kopell 1990) ...
- ... but possible if time-delay exists (Reddy, Sen, Johnston 1998)
Important implications for coupled cardiac cells, neurons, electronic circuits, financial cycles, etc.

Aim:

1. General understanding of stability in delayed networks

2. How does stability depend on coupling topology?
Coupled DDEs near Hopf bifurcation

\[ \dot{x}_k(t) = L_k x^t_k + \varepsilon f_k(x^t_k; \varepsilon) + \varepsilon \kappa g_k(x^t_1, \ldots, x^t_N; \varepsilon) \quad k = 1, \ldots, N \]

\[ x^t_k(\theta) = x_k(t + \theta) \in \mathbb{R}^n, \quad \theta \in [-\tau, 0] \quad \text{for} \quad x^t_k \in C = C([-\tau, 0], \mathbb{R}^n) \]

\[ L_k : C \to \mathbb{R}^n \text{ linear,} \quad \kappa \in \mathbb{R}, \text{ coupling strength} \]

\[ f_k, g_k \text{ are } C^2 \text{ and vanish at the origin,} \quad 0 < \varepsilon \ll 1 \]

Assume the linear problem (\( \varepsilon = 0 \)) has eigenvalues \( \pm i \omega_k \neq 0 \), all others eigenvalues with negative real parts.

Origin is an equilibrium, unstable for uncoupled system (\( \kappa = 0 \)).

What is the effect of coupling (\( \kappa \neq 0 \)) on stability?
Stability of coupled system

**Theorem 1** Let $\kappa \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following hold:

(i) If all the eigenvalues of the matrix $Q + \kappa P$ have negative real parts then the origin is asymptotically stable.

(ii) If $Q + \kappa P$ has an eigenvalue with positive real part then the origin is unstable.

Here, $Q$ and $P$ are complex $N \times N$ matrices, $Q$ is determined by the individual dynamics of units, and $P$ by the coupling functions.

$$\dot{z} = (Q + \kappa P)z$$
More precisely, let $F_k$ and $G_{kl}$ be $n \times n$ matrices, with elements of $BV$ on $[-\tau, 0]$

$$[D_1 f_k(0; 0)]\phi = \int_{-\tau}^{0} dF_k(\theta)\phi(\theta), [D_l g_k(0, \ldots, 0; 0)]\phi = \int_{-\tau}^{0} dG_{kl}(\theta)\phi(\theta)$$

$P = [p_{kl}] \in \mathbb{C}^{N \times N}$ is defined by

$$p_{kl} = tr \left[ \Psi_k^T(0) \int_{-\tau}^{0} dG_{kl}(\theta)\Phi_l(\theta) \right] + i tr \left[ J\Psi_k^T(0) \int_{-\tau}^{0} dG_{kl}(\theta)\Phi_l(\theta) \right]$$

if $\omega_k = \omega_l$, else $p_{kl} = 0$.

$Q = diag\{q_1, \ldots, q_N\}$ where

$$q_k = tr \left[ \Psi_k^T(0) \int_{-\tau}^{0} dF_k(\theta)\Phi_k(\theta) \right] + i tr \left[ J\Psi_k^T(0) \int_{-\tau}^{0} dF_k(\theta)\Phi_k(\theta) \right].$$
Corollaries

▶ If $g_k$ is independent of $x_k$ for all $k$, then $Q + \kappa P$ is an unstable matrix for any $\kappa \in \mathbb{R}$.
⇒ No death for “direct” coupling.

▶ Suppose the frequencies $\omega_1, \ldots, \omega_N$ are all distinct, and $\text{Re}(p_{kk})$ is negative (resp. positive) for all $k$. Then there exists $\kappa_0 > 0$ such that $Q + \kappa P$ is stable for all $\kappa > \kappa_0$ (resp. $\kappa < -\kappa_0$).
⇒ Frequency differences can cause oscillator death.

▶ If $\text{Re}(p_{kk})$ does not have the same sign for all $k$, or if $|\kappa| < \kappa_0$, then partial oscillator death may arise. (F.A., *Physica D*, 2003)
Diffusively-coupled identical oscillators

\[ \dot{x}_i(t) = L x_i^t + \varepsilon f(x_i^t; \varepsilon) + \varepsilon \kappa \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} \gamma(x_i^t, x_j^{t-\tau_0}) \]

\[ \gamma(\phi, \phi) = 0 \ \forall \phi \in \mathcal{C}, \ \text{e.g.} \ \gamma(\phi, \psi) = \psi - \phi \]

\[ \tau_0 = \text{transmission delay} \]

\[ A = [a_{ij}] = \text{(symmetric) adjacency matrix} \]

\[ d_i = \text{no. of neighbors of } i\text{th unit} \]
Graph Laplacian

- System viewed as a graph $\mathcal{G}$

- Symmetric neighborhood relation $i \sim j$

- $\mathcal{F}$ = the set of real-valued functions on $\mathcal{G}$

- Inner product $\langle u, v \rangle := \sum_{i=1}^{N} d_i u(i) v(i)$, norm $\|u\| := \langle u, u \rangle^{1/2}$ for $u, v \in \mathcal{F}$.

- Laplacian operator $\mathcal{L} : \mathcal{F} \to \mathcal{F}$ defined by $\mathcal{L}u(i) := \frac{1}{d_i} \sum_{j \sim i} (u(i) - u(j))$
Matrix form: $\mathcal{L} = I - D^{-1}A$, where $D = \text{diag}\{d_1, \ldots, d_N\}$ and $A$ is the adjacency matrix of $G$

**Fact.** $\mathcal{L}$ is self-adjoint and positive semidefinite. Its smallest eigenvalue is zero, and its largest eigenvalue $\lambda_{\text{max}}$ satisfies
\[
\frac{N}{N - 1} \leq \lambda_{\text{max}} \leq 2.
\]
Furthermore, $\lambda_{\text{max}} = N/(N - 1)$ if and only if $G$ is a complete graph of $N$ vertices, and $\lambda_{\text{max}} = 2$ if and only if $G$ is bipartite.
Bipartite graphs
Topology, delays, and stability

**Theorem 2** If $H > 0$, then there exist $\kappa \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that the origin is asymptotically stable for all $\varepsilon \in (0, \varepsilon_0)$. If $H < 0$ and $\kappa \in \mathbb{R}$, then there exists $\varepsilon_0 > 0$ such that the origin is unstable for all $\varepsilon \in (0, \varepsilon_0)$.

$$H = (\cos \zeta - \cos(\zeta + \tau_0)) (\cos \zeta + (\lambda_{\text{max}} - 1) \cos(\zeta + \tau_0))$$

$\zeta = \text{arg}(p_{11})$ = The "projection angle" of the coupling effects onto center manifold. Independent of topology and $\tau_0$, depends on which variables are used in coupling.
(a) \( \lambda_{\text{max}} = 2 \),
(b) \( \lambda_{\text{max}} = 1.8 \),
(c) \( \lambda_{\text{max}} \to 1 \)
No stability for $\tau_0 = 0$.
⇒ Stability results only from transmission delays.

$\lambda_{\text{max}}$ completely characterizes the effect of network topology. Smaller $\lambda_{\text{max}}$ implies larger stability region. (F.A., *J. Diff. Eq.*, in press)

All bipartite graphs have the same stability region, independent of size. Their stability region constitutes a lower bound for any graph.

Large complete graphs have the largest stability region among all graphs.

If $\tau_0$ is a distributed rather than a discrete delay, then stability regions are enlarged. (F.A., *Phys. Rev. Lett.* 2003)
Directed graphs and edge-dependent delays

\[ \dot{x}_i(t) = Lx_i^t + \varepsilon f(x_i^t; \varepsilon) + \varepsilon \kappa \frac{1}{d_i} \sum_{j \rightarrow i} \gamma(x_i^t, x_j^{t-\tau_{ji}}) \]

\( j \rightarrow i \): a directed edge from \( j \) to \( i \).

\( d_i \) = in-degree of vertex \( i \).

Delay-adjacency matrix \( \bar{A}_\tau \) defined by the elements

\[ \bar{a}_{ij} = \begin{cases} \exp(i \tau_{ji}) & \text{if} & j \rightarrow i \\ 0 & \text{otherwise} \end{cases} \]
Theorem 3  Let \( \{\rho_i\} \) be the set of eigenvalues of \( P = p_{11}(I - D^{-1}A_T) \).

1. If \( \text{Re}(\rho_i) \) are nonzero and have the same sign for all \( i \), then there exist \( \kappa \in \mathbb{R} \) and \( \varepsilon_0 > 0 \) such that the origin is asymptotically stable for all \( \varepsilon \in (0, \varepsilon_0) \).

2. If there is a pair \( i, j \) such that \( \text{Re}(\rho_i) \text{Re}(\rho_j) \leq 0 \) and \( \kappa \in \mathbb{R} \), then there exists \( \varepsilon_0 > 0 \) such that the origin is unstable for all \( \varepsilon \in (0, \varepsilon_0) \).

3. Special case: \( p_{11} \in \mathbb{R}\{0\} \). Then the origin is stable for some \( \kappa \in \mathbb{R} \) and \( 0 < \varepsilon \ll 1 \) if and only if \( D - A_T \) is nonsingular.
Summary

A general class of functional differential equations

Oscillatory activity through supercritical Hopf bifurcation

\[ \Downarrow \ (\text{Averaging - Center manifold}) \Downarrow \]

Finite dimensional problem

- For undirected networks, effect of connection topology on stability is completely characterized by the largest eigenvalue of the graph Laplacian.

- For directed networks and multiple delays, stability is characterized through the spectrum of a certain complex matrix.

- Similar results hold for coupled discrete-time systems.