Transverse Instability of Granular Avalanches

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“large conglomerates of discrete macroscopic particles”

Jaeger, Nagel, & Behringer, Rev. Mod. Phys. 1996
Kadanoff, Rev. Mod. Phys. 1999
de Gennes, Rev. Mod. Phys. 1999

- non-solid
  - no tensile stresses
- Granular Matter
- non-liquid
  - critical slope
- non-gas
  - inelastic collisions
Typical (Large-Scale) Avalanche Experiments
Laboratory-Scale Granular Avalanches

Daerr & Douady, Nature, 1999 (ENS)

- No flow
- Spontaneous avalanching
- Uphill avalanches
- Bistability
- Downhill avalanches
Triangular vs Uphill Avalanches

$h/d = 4, \phi = 31.5^\circ$

Experiment

Triangular (down-hill)

Theory

Balloon (up-hill)
Experimental set-ups

Fontainebleau sand
\[ d = 300 \, \mu m \]

Aluminium oxyde powder
\[ d = 15, 20, 30, 40 \, \mu m \]
Profile of the propagated wave

solitary wave: stable and stationary
Transverse instability of avalanches

F. Malloggi, E. Clement, B. Andreotii, 2005 submitted

Region III : $\theta_a < \theta$

Typical fing patterns in fluids/fluids with particles
Evolution of submarine and dry avalanches

Underwater avalanches

Dry sand avalanches (difference of two images)

Fingering is less pronounced for dry avalanches
No apparent instability for monodisperse glass beads
Initial instability from Prepatterened State

\[ \theta = 35^\circ \]
\[ d = 300 \mu m \]

no modulated initial conditions  modulated initial conditions
Linear instability

\( \theta = 35^\circ \)

\( d = 300\mu m \)

\( \text{air} \)

Maximum growth rate: \( \sigma_m \sim 2.5\,s^{-1} \) for \( \lambda_0 \sim 4\,\text{cm} \)
Early Experiments

- Fingering for rough sand
- No fingering for glass beads
- Conclusion: instability due to size segregation?? (was challenged later)

Theoretical model

Momentum equation

\[ \rho_0 \frac{Dv}{Dt} = -\nabla \cdot \sigma + \rho_0 g \]

\[ \text{div } v = 0 \]

\( \rho_0 \) - density of material (\( \rho_0 = 1 \))

\( g \) - gravity acceleration

\( v \) - hydrodynamic velocity

\( \frac{D}{Dt} = \partial / \partial t + v \nabla \) - material derivative

\( \sigma \) - stress tensor

Main fundamental problem:
Constitutive relation for stress valid for gran solid and liquid
Stress-stain relation for partially fluidized granular flow

\[ \sigma_{ij} = \sigma_{ij}^f + \sigma_{ij}^s \]

Here \( \sigma_{ij}^f \) - fluid part

\( \sigma_{ij}^s \) - quasistatic (contact) part

\[ \sigma_{ij}^f = -p_f \delta_{ij} + \mu_f \dot{\gamma}_{ij} \]  
Newtonian “fluid”

\[ \sigma_{ij}^s \]  
Anisotropic elastic “solid”
Order parameter $\rho$ for partially fluidized granular flows

$\rho$ characterizes the “state” of the granular matter:

$\rho = 0$ - pure fluid $\quad \rho = 1$ - pure solid

$$\sigma_{xy}^f = q(\rho)\sigma_{xy} \quad ; \quad \sigma_{xy}^s = (1 - q(\rho))\sigma_{xy}$$

$$\sigma_{ii}^f = q_i(\rho)\sigma_{ii} \quad ; \quad \sigma_{ii}^s = (1 - q_i(\rho))\sigma_{ii}$$

$q \rightarrow 1$ for $\rho \rightarrow 0$

$q \rightarrow 0$ for $\rho \rightarrow 1$

The simplest choice:

$$q(\rho) = 1 - \rho \quad ; \quad q_i(\rho) = 1 - \rho$$
Equation for the order parameter

\[ \tau D \rho / Dt = -\frac{\delta F}{\delta \rho} \]

Ginzburg-Landau free energy for “shear melting” phase transition

\[ F = \int [l^2 (\nabla \rho)^2 + f(\rho)] d\mathbf{r} \]

\( \tau, l \) – characteristic time & length (material parameters)

\[ \tau \dot{\rho} = l^2 \nabla^2 \rho - (1 - \rho)G(\rho, \delta) \]

Two stable states: \( \rho = \rho_f \) and \( \rho = 1 \)

One unstable state \( \rho_u \)

\( \delta \) is a control parameter \( \delta = f(\sigma_{ij}) \)

\( \delta \) is shear temperature

\( \delta \) vs Mohr-Coulomb yield condition

\( \rho = 0 \) – liquid; \( \rho = 1 \) – solid
Set of Equations

Momentum Conservation

\[ \rho_0 \frac{Dv}{Dt} = \partial \sigma_{ij} + \rho_0 g \]

Constitutive relation for stress tensor

\[ \sigma_{ij} = \sigma_{ij}^f + \sigma_{ij}^s; \sigma_{ij}^s = \rho \sigma_{ij}; \sigma_{ij}^f = (1 - \rho) \sigma_{ij} \]

Order parameter equation

\[ D\rho / Dt = \nabla^2 \rho - \rho (1 - \rho) (\delta - \rho) \]

Normalized control parameter – “shear temperature”

\[ \delta = \frac{(\sigma_{xz} / \sigma_{zz})^2 - \phi_1^2}{\phi_2^2 - \phi_1^2} \]

+Boundary conditions
Molecular Dynamics Simulations

\[ \rho = \frac{Z_{st}}{Z} \]

- \( Z_{st} \) is the static coordination number: the number of long-term (\( >1.1 t_c \)) contacts per particle.
- \( Z \) is the total coordination number: the total number of contacts per particle.

2304 particles (48x48), \( \varepsilon = 0.82; \ \mu = 0.3; \ P_{\text{ext}} = 13.45, V_x = 24 \)
Thin granular layer on rough incline

Equilibrium conditions

\[ \sigma_{zz,z} + \sigma_{xz,x} = -g \cos(\varphi) \]
\[ \sigma_{xzz,z} + \sigma_{xx,x} = g \sin(\varphi) \]

Stresses:

\[ \sigma_{zz} = -g \cos(\varphi) \]
\[ \sigma_{xz} = g \sin(\varphi) \]
\[ \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{yz} = 0 \]
\[ |\sigma_{xz} / \sigma_{zz}| = \tan \varphi \]

Boundary conditions:

\[ \rho = 1 \quad \text{for } z=0 \text{ (bottom)} \]
\[ \partial_z \rho = 0 \quad \text{for } z=h \text{ (free surface)} \]

Control parameter (MC criterion):

\[ \delta = \frac{\tan \varphi - 2 \tan \varphi_1 + \tan \varphi_2}{2(\tan \varphi_2 - \tan \varphi_1)} \]

\((\varphi_1, \varphi_2 \text{ - static/dynamic repose angles})\)
Single mode approximation

Close to the stability boundary \( \rho = 1 - A(x, y, t) \sin(\pi z / 2h) \)

\[
A_t = \partial_x^2 A + \partial_y^2 A + \left( \delta - 1 - \frac{\pi^2}{4h^2} \right) A + \frac{8(2 - \delta) A^2}{3\pi} - \frac{3}{4} A^3
\]

\[
h_t = - \nabla J \equiv - \alpha \partial_x \left( h^3 A \right) + \frac{\alpha}{\tan(\phi)} \nabla \left( h^3 A \nabla h \right)
\]

A-fluidization parameter, \( h \) – local thickness of the layer

Here \( \delta = \delta_0 - \beta h_x \)

\[
\beta = \frac{1}{2(\tan \phi_2 - \tan \phi_1)}
\]

\[
\alpha = \frac{2(\pi^2 - 8) g \sin \phi l^2}{\pi^3 \eta}
\]

- advection (transport) coefficient

\( \eta \) - granular viscosity (in fluid phase)
Soliton-like Avalanches

One-parametric family of solutions

- Avalanche velocity $V$ depends on trapped mass $m$

$$m = \int_{-\infty}^{\infty} (h(x) - h_0) \, dx$$

- $V$ increases with $m$
- There is a critical size of avalanche
Growth-rate of transverse instability

Numerical stability analysis

\( q \) – trans modulation wavenumber

\( \lambda(q) \) – linear growth rate

Experimental results

\( \lambda(q) \sim q \quad \text{for} \quad q \to 0 \)

(slope obtained analytically)

Instability disappears for \( \alpha < \alpha_c \)
Transverse instability of gran avalanches
Mechanism of Instability

- Velocity increases with mass
- Front bulging
- Sand flows down the bulge
- Further increase of velocity
Analytical Results

- Weakly-curved soliton avalanche

\[ A(x, y, t) = \bar{A}(x - x_0(y, t)); h(x, y, t) = \bar{h}(x - x_0(y, t)) \]

- Eqns for position \( x_0(y, t) \) and local mass \( m(y, t) \)

\[
\partial_t x_0 = V(m) + \partial_y^2 x_0 + ... \\
\partial_t m = -\xi \partial_y^2 x_0 + ..., \xi = \frac{\alpha}{\phi} \int_{-\infty}^{\infty} \bar{A} \bar{h}^3 \partial_x \bar{h} dx \\
\partial_t^2 x_0 = -\xi V'(m_0) \partial_y^2 x_0 + ... - \text{elliptic instability} \\
\]

- Transverse instability \( x_0, m \sim \exp(\lambda t + iqy) \)

\[
\lambda = |q| \sqrt{\xi V'(m_0)} + O(q^2)
\]
Conclusions

- OP description captures transverse instability
- Simple generic mechanism of instability: increase of mass → increase of velocity and fluidization → bulging of the front → increase of flux towards the bulge → …
- Impressive agreement with experiment
- Interesting connection with flows of thin fluid films (fingering without surface tension)

Typical fing patterns in fluids/fluids with particles