Eulerian and Lagrangian Time Scales in Turbulence

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December 14, 2005
Homogeneous and isotropic turbulence

Flow behind a grid (Van dyke, 1982)
\( \log[S_p(r)] \) vs \( \log(r) \)

\[
\delta u_\parallel(r, t) \equiv [u(x + r, t) - u(x, t)] \cdot \frac{r}{r}
\]

\[
S_p(r) \equiv \langle [\delta u_\parallel(r, t)]^p \rangle \sim r^{\zeta_p}
\]
(The She-Leveque formula gives good agreement with experimental results.)

- $\zeta_p = p/3$ (Kolmogorov 1941)

- Experimental and Direct Numerical Simulations (DNS) show that the function $\zeta_p$ versus $p$ is a nonlinear (convex) function of $p$. This phenomenon is called *Multiscaling*
Dynamic Scaling

\[ F_2(r, t) \equiv \langle \delta u(r, 0) \delta u(r, t) \rangle \]

According to K41, \( F_p(r, t) \) decays in time with characteristic time scale \( \tau_p(r) \sim r^{z_p} \) with \( z_p = 2/3 \). (Simple dynamic scaling)

\( z_p \) is the dynamic scaling exponent. (Note the similarity with dynamic scaling as defined in critical point phenomenon)
Sweeping Effect

- Small eddies are advected by large eddies (swept) without any significant change to the small eddies (Taylor hypothesis).
- Also the non-linear term of the Navier–Stokes eqn. couples the high fourier modes to the smallest ones.
- Hence typical time and length scales are linearly related.
- Dynamic scaling exponent $z = 1$.
- The K41 prediction applies to the Lagrangian velocity.
Quasi-Lagrangian transformation

$$u^{ql}(r, t|r_0, t_0) = u[r + \rho(t|r_0, t_0)]$$

(Belinicher and L’vov 1988)

- Quasi-Lagrangian velocity is the Eulerian velocity relative to a fluid particle. This is expected to show no sweeping. The equal-time behaviour is similar to Eulerian and dynamic behaviour is similar to the Lagrangian velocities.
- The sweeping effect is removed by the quasi-Lagrangian transformation!
Questions

- How to remove the sweeping?
- Is the dynamic scaling for Lagrangian/quasi-Lagrangian velocities simple or is there dynamic multiscaling?
- Is there a model for which analytic calculations illustrating dynamic (multi)scaling is possible?
\( S^\theta_p(r) \equiv \langle [\delta \theta(r)]^p \rangle \sim r^{\zeta_p^\theta} \)

\( F_2^\theta(r) \equiv \langle \delta \theta(r, 0) \delta \theta(r, t) \rangle \)
Kraichnan model of passive scalar

\[ \partial_t \theta + (u \cdot \nabla) \theta = \kappa \nabla^2 \theta + f^\theta, \]

Velocity obeys Navier–Stokes equation:

\[ \partial_t u + (u \cdot \nabla) u = \nu \nabla^2 u + \nabla p/\rho + f/\rho \]

Velocity is random, Gaussian with co-variance

\[ \langle u_i(x, t) u_j(x + \ell, t') \rangle = 2D_{ij}(\ell) \delta(t - t') \]

\[ D_{ij}(\ell) = D_0 \delta_{ij} - \frac{1}{2} d_{ij}(\ell) \]

\[ \begin{align*}
D_0 &\sim L^\xi \\
L &\to \infty \text{ and } \eta \to 0, \quad d_{ij} = D_1 \ell^\xi \left[ (d - 1 + \xi) \delta_{ij} - \frac{\xi \ell_i \ell_j}{\ell^2} \right].
\end{align*} \]
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\[ L \to \infty \text{ and } \eta \to 0, \quad d_{ij} = D_1 \ell^\xi \left[ (d - 1 + \xi) \delta_{ij} - \xi \frac{\ell_i \ell_j}{\ell^2} \right]. \]
Equal-time statistics

- Due to the white-in-time nature of velocity we can get closed equation of motion for velocity moments.
- Multiscaling can be analytically (but perturbative) demonstrated.
- \( S_p(\ell) \sim \ell^{\xi_p} \).
- \( 0 < \xi < 2 \).
- Structure functions have good limits, not correlation functions.
Dynamic structure functions

\[ \delta \phi(x, t, r) \equiv \phi(x + r, t) - \phi(x, t) \]
\[ \mathcal{F}_2^\phi(r, t) = \langle [\delta \phi(x, 0, r) \delta \phi(x, t, r)] \rangle \]
\[ = 2C^\phi(0, t) - 2C^\phi(r, t) \]
\[ C^\phi(r, t) \equiv \langle \phi(x + r, 0) \phi(x, t) \rangle \]

- \( t \) strictly positive.
- \( \phi \) can be either Eulerian or quasi–Lagrangian quantity.
Dynamic scaling

\[ \partial_t C^\phi(r, t) = \langle \phi(x + r, 0) \partial_t [\phi(x, t)] \rangle \]
\[ = -\langle \phi(0)(u \cdot \nabla)\theta \rangle + \kappa \nabla^2 \langle \phi(0)\phi \rangle + \langle \phi(0)f^\phi \rangle \]

\[ \partial_t C^\theta(r, t) = D^0(L) \partial_{ij} C^\theta \sim L^\xi \partial_{ij} C^\theta; \]
\[ \partial_t C^\hat{\theta}(r, t) = (D^0 \delta_{ij} - D_{ij}) \partial_{ij} C^\hat{\theta} \sim d_{ij}(r) \partial_{ij} C^\hat{\theta}. \]
\[ C^\phi(r, t) \sim \exp[-t/\tau^\phi(r)] \]

\[ \tau^\hat{\theta}(r, t) = r^{1-\xi}; \ \tau^\theta(r, t) = r^2 \]

In the limit of \( L \to \infty \), \( C^\theta(r, t) \) diverges for all \( r \).
\( z^{ql} = 2 - \xi \), a bridge relationship.
Dynamic scaling

$$\partial_t C^\phi(r, t) = \langle \phi(x + r, 0) \partial_t [\phi(x, t)] \rangle$$
$$= -\langle \phi(0) (u \cdot \nabla) \theta \rangle + \kappa \nabla^2 \langle \phi(0) \phi \rangle + \langle \phi(0) f \phi \rangle$$

$$\partial_t C^\theta(r, t) = D^0(L) \partial_{ii} C^\theta \sim L^\xi \partial_{ii} C^\theta;$$
$$\partial_t C^{\hat{\theta}}(r, t) = (D^0 \delta_{ij} - D_{ij}) \partial_{ij} C^{\hat{\theta}} \sim d_{ij}(r) \partial_{ij} C^{\hat{\theta}}.$$  
$$C^\phi(r, t) \sim \exp[-t/\tau^\phi(r)]$$

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Dynamics of Kraichnan model

- Generalization to higher order is possible but messy.
- Eulerian time-dependent structure functions have no good limit due to sweeping.
- Quasi-Lagrangian time-dependent structure functions remain finite, shows exponential decay in time, and simple dynamic scaling:
  \[ z^{ql} = 2 - \xi \]
- Only analytical result for time-dependent structure functions in any form of turbulence.
Dynamics of fluid turbulence

- No known analytical attack.
- We perform $512^3$ spectral DNS to calculate time-dependent structure function but to get well averaged result takes too long.
- An extension of the multifractal model provides illuminating bridge-relations.
Varieties of time-scales

\[ T_{p,1}^I(r) \equiv \frac{1}{S_p(r)} \int_0^\infty F_p(r, t) \, dt \sim r^{\frac{z_p}{p}^I} \]

\[ T_{p}^{D,2} \equiv \frac{1}{S_p(r)} \frac{\partial^2 F_p(r, t)}{\partial t^2} \sim r^{\frac{z_p}{p}^D} \]
Varieties of dynamic multiscaling

For Lagrangian or quasi-Lagrangian velocities:

\[ z_{p,1}^L = 1 + [\zeta_{p-1} - \zeta_p], \]
\[ z_{p,2}^D = 1 + [\zeta_p - \zeta_{p+2}] / 2. \]
For the full passive-scalar problem:

\[ z_{p,2}^D = 1 - \frac{\zeta_2^u}{2} \]

\[ z_{p,1}^I = 1 - \zeta_{-1}^u \]
Conclusion

- Kraichnan model shows simple dynamic scaling (although equal-time structure functions show multiscaling)
- Dynamic multiscaling implies "breakdown of simple dynamical scaling"
- A consequence of the multifractality.
- Bridge relations are numerically verified in GOY shell model.
- To numerically obtain bridge relations in Navier–Stokes equation we have to
  - Remove sweeping effect.
  - Calculate dynamic structure functions
  - Extract the dynamic scaling exponents
