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Towards a spectral approach to ageing

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Plan

1. Ageing.
2. The REM-like trap model.
3. Sinai’s random walk.
Ageing

**Setting:** Markov processes, $X_t$, on some state-space $S$ with generator $\mathcal{L}$. In this talk I will always consider the setting of a reversible Markov chain with reversible measure $\mathcal{Q}$, so $\mathcal{L}$ is self-adjoint.  

**General question:** Relation between long-time properties of the dynamics and spectral properties of $\mathcal{L}$.

**Dynamic types:**

- **Rapid mixing:** Process converges to equilibrium exponentially fast.
- **Spectral signature:** Spectral gap between eigenvalue $0$ and rest of spectrum.

- **Metastability:** State space decomposes into “quasi-invariant” sub-spaces; multiple time-scales: rapid mixing within quasi-invariant subspace at “short” times, transitions between quasi-invariant subspace at “large” time-scale.
- **Spectral signature:** finite set of “exponentially small” followed by spectral gap. Small eigenvalues correspond to inverse exit times from metastable states.

- **Ageing:** Systems are neither mixing nor metastable, but show slow (power-law) transients towards equilibrium.
- **Spectral signature:** Subject of this talk.
Consider a (stochastic) process $X_t$. Define a correlation function

$$f(t_w, t) = C(X_{t_w}, X_{t_w+t})$$

General definition: $X_t$ ages, if $f(t_w, t)$, for $t_w, t$ large, depends on $t_w$. More restrictive: $f(t_w, t)$ is a function of $t/t_w$. [sometimes of $t^\theta/t_w$ (sub/super-ageing)].

This depends on the particular choice of correlation function. A more ”intrinsic” characterisation would maybe nicer.

For a system to age, at time $t_w$, it should be (typically) found in a states with typical “reaction time” $T(t_w)$. 
A tentative scenario (clearly not sufficiently general)

For $A \subset S$, let $\bar{\lambda}(A)$ denote the smallest eigenvalue of the generator $\mathcal{L}$ with Dirichlet conditions on $A^c$.

Note: If $0 \in A$, then
\[
\mathbb{P}_0 [X_t \in A] \sim e^{-t \bar{\lambda}(A)}
\]

Find an increasing family of sets, $0 \in S_i \subset S$, such that, for all $i$,
\[
\begin{align*}
\triangleright & S_i \subset S_{i+1} \\
\triangleright & \bar{\lambda}_i \equiv \bar{\lambda}(S_i) > \bar{\lambda}(S_{i+1}) \equiv \bar{\lambda}_{i+1} \\
\triangleright & \mathbb{Q}(S_{i-1} | S_i) \sim 0.
\end{align*}
\]

Then, roughly, at times $\bar{\lambda}_i^{-1} < t < \bar{\lambda}_{i+1}^{-1}$, $X_t$ should be distributed as $\mathbb{Q}(\cdot | S_i)$, and hence localised within $S_i \setminus S_{i-1}$.

$S_i \setminus S_{i-1}$ will in many examples decompose into smaller subsets, $S_{i,k}$, with $\bar{\lambda}(S_{ik})$ comparable to $\bar{\lambda}(S_i)$.

Can one translate such a picture directly into statements about the spectrum of $\mathcal{L}$?
References:

- **General and trap models:**


  C. Montus, Ma-Dasgupta renormalization studies of various disordered systems, 2004
The REM like trap model

Trap models:
\[ \mathcal{G} = (\mathcal{S}, \mathcal{E}) \text{ finite graph.} \]

\[ \mathcal{E} \equiv \{ E_i, i \in \mathcal{S} \} \text{ iid r.v.} \]

\( Y(t) \) continuous–time random walk on \( \mathcal{G} \) with \( \mathcal{E} \)–dependent transition rates, \( c_{i,j} \)

REM-like trap model:
\( \mathcal{S} \equiv \{1, \ldots, N\} \), \( \mathcal{G} \) complete graph.

\[ c_{i,j} = x_i \equiv e^{-E_i}/N, \text{ for } i \neq j \]

\( E_i \) iid exp. with parameter \( 0 < \alpha < 1 \).

\[ \mathcal{L}_N \equiv \begin{pmatrix}
\frac{(N-1)x_1}{N} & -\frac{x_1}{N} & \cdots & -\frac{x_1}{N} \\
-\frac{x_2}{N} & \frac{(N-1)x_2}{N} & \cdots & -\frac{x_2}{N} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{x_N}{N} & -\frac{x_N}{N} & \cdots & \frac{(N-1)x_N}{N}
\end{pmatrix} \]
Correlation function:

\[ \Pi_N(t, t_w) \equiv \mathcal{P}_N \left( Y_N(s) = Y_N(t_w), \forall s \in [t_w, t_w + t] \right). \]

Ageing: (Bouchaud-Dean ’92) For almost all \( E \), and for all \( \theta > 0 \),

\[
\lim_{t_w \uparrow \infty} \lim_{N \uparrow \infty} \Pi_N(\theta t_w, t_w) = \frac{\sin(\pi \alpha)}{\pi} \int_{\theta \over 1+\theta}^{1} u^{-\alpha} (1 - u)^{\alpha-1} du
\]

which is a somewhat generic behaviour for many ageing systems.
Proposition 1. Eigenvectors, $0 = \lambda_1 < \lambda_2 \cdots < \lambda_N$, of $\mathcal{L}_N$ are the zeros of

$$\phi(\lambda) \equiv \sum_{j=1}^{N} \frac{\lambda}{x_j - \lambda}$$

For all $i$, $x_i < \lambda_{i+1} < x_{i+1}$

The $j$-th eigenvector, $\psi^{(j)}$, has components

$$\psi^{(j)}_i \equiv \frac{x_j}{x_j - \lambda_i}$$

Corollary. Spectral distribution

$$\sigma_N \equiv N^{-1} \sum_{j=1}^{N} \delta_{\lambda_j} \to \alpha x^{\alpha-1} \, dx \text{ on } \mathbb{R}_+, \text{ a.s.}$$
Proposition 2. \textit{Eigenvalues}, $0 = \lambda_1 < \lambda_2 \cdots < \lambda_N$, of $\mathcal{L}_N$ are the zeros of

$$\phi(\lambda) \equiv \sum_{j=1}^{N} \frac{\lambda}{x_j - \lambda}$$

\textit{For all} $i$, $x_i < \lambda_{i+1} < x_{i+1}$\textit{.} The $j$-th eigenvector, $\psi^{(j)}$, has components

$$\psi^{(j)}_i \equiv \frac{x_j}{x_j - \lambda_i}$$

Corollary. Spectral distribution $\sigma_N \equiv N^{-1} \sum_{j=1}^{N} \delta_{\lambda_j} \rightarrow \alpha x^{\alpha-1} dx$ on $\mathbb{R}_+$, a.s.
Implications of spectral results.

Consider the process as a process on the “waiting times”, i.e. set $x_N(t) \equiv x_{Y_N(t)}$. For any bounded function $h$, we can represent

$$E_N(h(x_N(t))) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\gamma_k e^{-\lambda_k t}}{x_j - \lambda_k} h(x_j)$$

where $\gamma_k \equiv \|\phi^k\|_2^2 = \sum_{j=1}^{N} \frac{x_j}{(x_j - \lambda_k)^2}$.

Key technical result:

$$E_N(h(tx_N(t)) \to \frac{1}{2\pi i} \int_{\gamma} \frac{e^{-t\lambda} \int_0^1 \frac{h(xt)}{\lambda-x} x^{\alpha-1} dx}{\lambda} \int_0^1 \frac{1}{\lambda-x} x^{\alpha-1} dx} d\lambda, \text{ a.s.}$$

That is, the random variable $tx_N(t)$ or $t/\tau_N(t)$ converges to a random variable whose distribution is explicitly computed. All ageing results derive easily from here.
Sinai’s random walk

\( X_n \) discrete time random walk on \( \mathbb{Z} \), transition probabilities

\[
P(X_{n+1} = x + 1 \mid X_n = x) = \omega_x \quad P(X_{n+1} = x - 1 \mid X_n = x) = 1 - \omega_x.
\]

\( 0 < a < \omega_x < b < 1 \) iid random variables s.t. \( \mathbb{E} \left( \ln \left( \frac{\omega_x}{1-\omega_x} \right) \right) = 0. \)

Well-know facts:

**Localisation:** [Sinai ’82]

There is random process, \( m^{(n)}(\omega) \), depending only on the \( \omega_x \), s.t.

\[
\frac{X_n}{\ln^2 n} - m^{(n)} \to 0 \quad \text{in} \ P_0-\text{probability},
\]

**Ageing:** [Dembo, Guionnet, Zeitouni ’01]

\[
\lim_{n \uparrow \infty} P_0 \left( X_n \sim X_{nh} \right) = h^{-2} \left( \frac{5}{3} + \frac{2}{3} e^{-(h-1)} \right)
\]
Results

As is well-known, Sinai’s walk is a random walk in a potential, i.e. invariant measure is

\[ \mu(x) = \exp(-V(x)) \]

with

\[ V(x) = \begin{cases} 
\sum_{i=1}^{x} \ln \frac{1-\omega_i}{\omega_i}, & \text{if } x \geq 1, \\
0, & \text{if } x = 0, \\
-\sum_{i=x+1}^{0} \ln \frac{1-\omega_i}{\omega_i}, & \text{if } x \leq -1.
\end{cases} \]  

(1)

Rescaling: We will consider the process confined to a box \([-N, N]\). It is convenient to rescale to a unit box \([-1, 1]\) with spacing \(1/N\). The invariant measure is then

\[ \mu(x) = \exp\left(-\sqrt{N}V(N)(x)\right) \]

where, for \(N\) large, \(V(N)\) can be strongly approximated by a Brownian motion, in the sense that

\[ P^{(N)}\left(\sup_{x \in [-1,1]} \left| V^{(N)}(x) - B_x \right| > \frac{C_1 \ln N}{\sqrt{N}} \right) < \frac{C_2}{N^{C_3}}. \]  

(2)

[Komlós-Major-Tusnády]
**h-extrema**

If \( g \) is any continuous function on \([-1, 1]\), we call \( x \in [-1, 1] \) an \( h\)-minimum (for \( \gamma \)), if there exist \( a, b \in [-1, 1] \) with

\[
a < x < b, \quad \gamma(a) \geq \gamma(x) + h, \quad \gamma(b) \geq \gamma(x) + h, \quad \text{and} \quad \gamma(x) = \min_{[a,b]} \gamma.
\]

The absolute maxima between two consecutive \( h\)-minima, \( x, y \), are called \( h\)-maxima, or saddle-points, \( z^*(x, y) \).
Relevant $h$-minima

Consider now $V^{(N)}$ on $[-1, 1]$. For fixed $h$, we call $M_h^- \equiv \{x_1, \ldots, x_n\}$ the set of $h$-minima. The intervals between the consecutive $h$-maxima are called $h$-valleys.

Order $M_h^-$ in such a way that, if

$$S_{h,k} \equiv \{-1, 1, x_1, \ldots, x_k\}, \quad 0 \leq k \leq n$$

then, for some $\delta > 0$,

$$V^{(N)}(z(x_k, S_{h,k-1})) - V^{(N)}(x_k) \geq \max_{n \geq j > k} \left( v^{(N)}(z(x_j, S_{h,j-1})) - V^{(N)}(x_j) \right) + \delta$$
Small eigenvalues

**Theorem 3.** Given $h, \delta > 0$, with probability tending to one the following holds: 
$1 \leq q = |M_h^−|$, and if $λ∗_N$ denotes the principal eigenvalue of the operator 
$L^{(N)}(I_N \setminus M_h^−(V_N))$, then

$$\sigma(L_N) \cap [0, \lambda∗_N) = \left\{ \lambda_1^{(N)} < \lambda_2^{(N)} < \cdots < \lambda_q^{(N)} \right\}$$

and

$$\lambda_k^{(N)} \leq c(κ)N^2e^{-δN}λ_k^{(N)} \quad ∀k = 1, \ldots, q - 1$$

$$\lambda_q^{(N)} \leq c(κ)e^{-δN}λ∗_N$$

$$λ∗_N \geq N^{-2}e^{-h\sqrt{N}}$$

Moreover, for $1 \leq k \leq q$,

$$c(κ)N^{-2}\exp \left\{ \sqrt{N} \left[ V_N(z^*(x_k, S_{h,k-1})) - V_N(x_k) \right] \right\} \leq \lambda_k^{(N)} \leq c′(κ)\exp \left\{ \sqrt{N} \left[ V_N(z^*(x_k, S_{h,k-1})) - V_N(x_k) \right] \right\}$$

[in agreement with results of LeDoussal and Monthus using Ma-Dasgupta RG]
**Theorem 4.** For each \(1 \leq k \leq q\), the simple eigenvalue \(\lambda_k^{(N)}\) has eigenvector \(\psi_k^{(N)}\) that satisfies

\[
\left\| \psi_k^{(N)} - \frac{h_{x_k,S_{k-1}}}{\|h_{x_k,S_{k-1}}\|_2} \right\|_2 \leq e^{-\delta \sqrt{N}}
\]

where

\[
h_{x_k,S_{k-1}}(x) \equiv \mathcal{P}_x (\tau_{x_k} < \tau_{S_{k-1}})
\]
Consequences: Localisation

Define the $\ln n$ valley covering the origin can be written as $(a^{(n)}, m^{(n)}, b^{(n)})$.

Set $m^{(n)}(\omega) \equiv m^{(n)}(\omega) \ln^2 n$.

$A_n \equiv (a^{(n)}, b^{(n)}) \cap \mathbb{Z}, \quad D_n \equiv ((m^{(n)} - 2\delta_n) \ln^2 n, (m^{(n)} + 2\delta_n) \ln^2 n) \cap A_n$, where $\delta_n \gg \sqrt{\ln \ln n}$.

$$
P^\omega_0(X_n \in D_n) \geq P^\omega_0(X_n \in D_n, X_k \in A_n \forall 0 \leq k \leq n)$$

$$= \frac{1}{\mu(0)} (1, (1 - \mathcal{L}(A_n))^n 1_{D_n})$$

$$= \sum_{j=1}^{\left|A_n\right|} \left(1 - \lambda_j^{(n)}\right)^n \left(\psi_j^{(n)}, 1_{D_n}\right) \psi_j^{(n)}(0)$$

But: by the choice of $A_n$, all terms in the sum but the first vanish, and the first tends to one. This gives a refinement of Sinai’s theorem.
Why choose $A_n$?

Obviously, if we choose a smaller interval, we get a lower bound that is too small! But we could choose a bigger one, $A \supset A_n$. Then we have still the same expression with $A_n$ replaced by $A$, and the sum may involve even smaller eigenvalues.

Consider the particular case where $n$ and $n' > n$ the smallest value such that $m^{(n')} \neq m^{(n)}$. Let $A = A_{n'}$. Assume that there are just two eigenvalues of $\mathcal{L}(A_{n'})$ smaller than $n^{-1}$. Then

$$ P^\omega_0 (X_n \in D_{n'}) \geq \sum_{j=1}^{2} \left( \psi_j^{(n')}, 1_{D_{n'}} \right) \psi_j^{(n')} (0) \sim 0 $$

but also

$$ \left( \psi_1^{(n')}, 1_{D_{n'}} \right) \psi_1^{(n')} (0) \sim 1 $$

and thus

$$ \left( \psi_2^{(n')}, 1_{D_{n'}} \right) \psi_2^{(n')} (0) \sim -1 $$
Now focus on times that are of the order of an inverse eigenvalue, $t/\lambda_2^{(n')}$. Then

$$P_0^\omega \left( X_{t/\lambda_2^{(n')}} \in D_{n'} \right) \sim \left[ \left( \psi_1^{(N)}, 1_{D_{n'}} \right) \psi_1^{(n')} (0) + e^{-t} \left( \psi_2^{(n')}, 1_{D_{n'}} \right) \psi_2^{(N)} (0) \right]$$

$$\sim 1 - e^{-t}$$

Since there is an infinity of values $n'$ that allow for this construction, there is an infinite of increasing random time-scales on which the process is exiting exponentially, a metastable state towards a more stable one.