A Hierarchy of Automatic $\omega$-Words having a decidable MSO Theory

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An $\omega$-word over $\Sigma$ is a function $w : \mathbb{N} \rightarrow \Sigma$, alternatively represented by the word structure $(\mathbb{N}, <, \{w^{-1}(a)\}_{a \in \Sigma})$. 

Theorem (cf. Rabinovich, Thomas ‘06)

The MSO theory of $W$ is decidable iff there is a recursive factorization $w = w_0 \cdot w_1 \cdot \ldots \cdot w_n \cdot \ldots$ such that for every morphism $\psi$ into a finite monoid $M$ the contraction of $w$ wrt. $\psi$ and $f$:

$$w_{\psi} f = \psi(w_0) \cdot \psi(w_1) \cdot \ldots \cdot \psi(w_n) \cdot \ldots \in M$$

$\omega$ is ultimately periodic (with both period and threshold computable from $\psi$).
ω-Words

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**Theorem (cf. Rabinovich, Thomas ’06)**

*The MSO theory of \( W_w \) is decidable iff there is a recursive factorization*

\[
w = w_0 \cdot w_1 \cdot \ldots \cdot w_n \cdot \ldots\]

\[
f(0) \quad f(1) \quad f(2) \quad f(n) \quad f(n+1) \quad \ldots\]
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$$w_f^\psi = \psi(w_0) \cdot \psi(w_1) \cdot \ldots \cdot \psi(w_n) \cdot \ldots \in M^\omega$$

is ultimately periodic (with both period and threshold computable from $\psi$).
Morphic words

An word \( w \in \Sigma^\omega \) is morphic if there is a morphism \( \tau : \Gamma^* \to \Gamma^* \) with \( \tau(a) = au \) for some \( a \in \Gamma \) and a morphism \( h : \Gamma^* \to \Sigma^* \) such that

\[
w = h(a \cdot u \cdot \tau(u) \cdot \tau^2(u) \cdot \ldots \cdot \tau^n(u) \cdot \ldots)\]

Example

Let \( \tau : a \mapsto ab, b \mapsto ba \). Its fixed point \( \tau^\omega(a) \) is the Prouhet-Thue-Morse sequence

\[
t = a \cdot b \cdot ba \cdot baab \cdot baababba \cdot \ldots\]

Example

Consider \( \tau : a \mapsto ab, b \mapsto ccb, c \mapsto c \) and \( h : a, b \mapsto 1, c \mapsto 0 \). Then \( \tau^\omega(a) = a \cdot b \cdot ccb \cdot ccc cb \cdot \ldots \) and \( h(\tau^\omega(a)) \) is the characteristic sequence of the set of squares.
Morphic words

An word $w \in \Sigma^\omega$ is *morphic* if there is a morphism $\tau : \Gamma^* \to \Gamma^*$ with $\tau(a) = au$ for some $a \in \Gamma$ and a morphism $h : \Gamma^* \to \Sigma^*$ such that

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\tau^\omega(a) = a \cdot b \cdot ccb \cdot ccccb \cdot c^6b \cdot \ldots
\]

and \( h(\tau^\omega(a)) \) is the characteristic sequence of the set of squares.
Deciding the MSO theory of morphic words
[Carton, Thomas '02]

Consider
\[ w = h(a \cdot u \cdot \tau(u) \cdot \tau^2(u) \cdot \ldots \cdot \tau^n(u) \cdot \ldots) \]
and a morphism \( \psi \) into a finite monoid \( M \).
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The contraction of \( w \) wrt. \( \psi \) and \( f_\tau \),

\[ w_{f_\tau}^\psi = \psi(h(a)) \cdot \psi(h(u)) \cdot \psi(h(\tau(u))) \cdot \psi(h(\tau^2(u))) \cdot \ldots \cdot \psi(h(\tau^n(u))) \cdot \ldots, \]
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is ultimately periodic, since there are (computable) \( N \) and \( p \) such that
\[ \psi \circ h \circ \tau^{n+p} = \psi \circ h \circ \tau^n \quad (n > N) \]
An **automatic presentation** of a word $w \in \Sigma^\omega$ comprises regular sets $D$ and $P_a$ ($a \in \Sigma$), a synchronized rational binary relation $\prec$ over some alphabet $\Gamma$, such that $(D, \prec, \{P_a\}_{a \in \Sigma}) \cong W_w$. In particular, $(D, \prec) \cong (N, <)$ is a regular weak numeration system.

**Facts**
- The FO\textsubscript{mod} theory of every automatic structure is decidable.
- The class of automatic structures is closed under FO\textsubscript{mod}-interpretations.

Vince Bárány (RWTH Aachen)
An **automatic presentation** of a word \( w \in \Sigma^\omega \) comprises

- regular sets \( D \) and \( P_a \ (a \in \Sigma) \),
- a synchronized rational binary relation \( \prec \)

over some alphabet \( \Gamma \), such that \((D, \prec, \{P_a\}_{a \in \Sigma}) \cong W_w\).

In particular, \((D, \prec) \cong (\mathbb{N}, <)\) is a regular *weak* numeration system.
Automatic presentations of $\omega$-Words

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regular sets $D$ and $P_a$ ($a \in \Sigma$),
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- The $\text{FO}^{\text{mod}}$ theory of every automatic structure is decidable.
- The class of automatic structures is closed under $\text{FO}^{\text{mod}}$-interpretations.
The usual choice for $<$ is the length-lexicographic ordering

$$x <_{\text{lex}} y \iff |x| < |y| \text{ or } |x| = |y| \text{ and } x <_{\text{lex}} y$$
Length-lexicographic presentations

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Proposition (Rigo, Maes ’02)

An $\omega$-word is morphic iff it is automatically presentable using $\prec_{\text{lex}}$. 
Morphisms of $k$ stacks

$k$-stacks as parenthesized words  or  as trees of height $k$

\[
[[abb][a][ba]]
\]

\[
\begin{array}{c}
a \\
b \\
a \\
\downarrow \\
b \quad a
\end{array}
\]
Morphisms of $k$ stacks

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Morphisms of $k$-stacks $\approx$ $k$-stack of morphisms:

$$\text{Stack}_{\Gamma}^{(0)} = \Gamma \quad \text{Hom}_{\Gamma}^{(0)} = \Gamma \rightarrow \Gamma$$

$$\text{Stack}_{\Gamma}^{(k+1)} = [(\text{Stack}_{\Gamma}^{(k)})^*] \quad \text{Hom}_{\Gamma}^{(k+1)} = [(\text{Hom}_{\Gamma}^{(k)})^*] \quad (\text{uniformity!})$$
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\]

\[
\begin{align*}
\text{Hom}_\Gamma^{(0)} &= \Gamma \rightarrow \Gamma \\
\text{Hom}_\Gamma^{(k+1)} &= \text{[(Hom}_\Gamma^{(k)})^*]\end{align*}
\] (uniformity!)

Application:

- $\varphi^{(0)}(\gamma^{(0)})$ is as given,
- for $\varphi^{(k+1)} = [\varphi_1^{(k)} \ldots \varphi_s^{(k)}]$ and $\gamma^{(k+1)} = [\gamma_1^{(k)} \ldots \gamma_t^{(k)}]$

\[
\varphi^{(k+1)}(\gamma^{(k+1)}) = [\varphi_1^{(k)}(\gamma_1^{(k)})\ldots\varphi_s^{(k)}(\gamma_1^{(k)}) \ldots \varphi_1^{(k)}(\gamma_t^{(k)})\ldots\varphi_s^{(k)}(\gamma_t^{(k)})]
\]
An word $w \in \Sigma^\omega$ is $k$-morphic if there is a morphism $\varphi \in \text{Hom}_\Gamma^{(k)}$ a $k$-stack $\gamma \in \text{Stack}_\Gamma^{(k)}$ and a homomorphism $h : \Gamma^* \rightarrow \Sigma^*$ such that

$$w = h \left( \prod_{n=0}^{\infty} \text{leaves}(\varphi^n(\gamma)) \right).$$
$k$-Morphic words

An word $w \in \Sigma^\omega$ is $k$-morphic if there is a morphism $\varphi \in \text{Hom}_k^\Gamma$ a $k$-stack $\gamma \in \text{Stack}_k^\Gamma$ and a homomorphism $h : \Gamma^* \rightarrow \Sigma^*$ such that

$$w = h \left( \prod_{n=0}^{\infty} \text{leaves}(\varphi^n(\gamma)) \right).$$

Example

Let $\gamma = [[\#]]$, $\varphi = [\varphi_0 \varphi_1]$ with $\varphi_i : \begin{array}{c|ccc} & 0 & \mapsto & 0 \\ \# & 1 & \mapsto & 1 \\ \# & \# & \mapsto & i\# \end{array}$ (Non-uniform!)

Similarly, $s = 12345678910111213 \ldots$ (Champernowne word) is 2-morphic.
Consider $u = a_0a_1 \ldots a_{tk-1} \in \Sigma^t$.
Its $k$-split is $(u^{(1)}, \ldots, u^{(k)})$ with $u^{(i+1)} = a_ia_{k+i} \ldots a_{(t-1)k+i}$ f.a. $i < k$.
Additionally, let $u^{(0)} = 1|u|$.
Conversely, $u = \otimes_k(u^{(1)}, \ldots, u^{(k)})$ is the $k$-shuffle of the $u^{(i)}$-s.
Consider $u = a_0a_1 \ldots a_{tk-1} \in \Sigma^t k$. Its $k$-split is $(u^{(1)}, \ldots, u^{(k)})$ with $u^{(i+1)} = a_ia_{k+i} \ldots a_{(t-1)k+i}$ f.a. $i < k$. Additionally, let $u^{(0)} = 1|u|$. Conversely, $u = \bigotimes_k (u^{(1)}, \ldots, u^{(k)})$ is the $k$-shuffle of the $u^{(i)}$-s.

For $0 \leq i < k$ we define the equivalence

$$u \equiv_i v \iff \forall j \leq i \quad u^{(j)} = v^{(j)}$$

(implying $|u| = |v|$).
Consider $u = a_0 a_1 \ldots a_{tk-1} \in \Sigma^t$. Its $k$-split is $(u^{(1)}, \ldots, u^{(k)})$ with $u^{(i+1)} = a_ia_{k+i} \ldots a_{(t-1)k+i}$ f.a. $i < k$. Additionally, let $u^{(0)} = 1|u|$. Conversely, $u = \otimes_k(u^{(1)}, \ldots, u^{(k)})$ is the $k$-shuffle of the $u^{(i)}$-s.

For $0 \leq i < k$ we define the equivalence

$$u =_i v \iff \forall j \leq i \ u^{(j)} = v^{(j)} \quad (\text{implying } |u| = |v|).$$

Consider some lin. ord. $<$ of $\Sigma$ with induced $\langle \text{lex} \rangle$. The induced $k$-length-lexicographic ordering $\langle k\text{-llex} \rangle$ is defined as

$$u <_{k\text{-llex}} v \iff |u| < |v| \lor \exists i < k : u =_i v \land u^{(i+1)} \langle \text{lex} \rangle v^{(i+1)}.$$
**Theorem**

*For all* $k$, *an* $\omega$-*word is* $k$-*morphic iff it has an aut. pres. using* $<_k$-lex. *"*
Theorem

For all $k$, an $\omega$-word is $k$-morphic iff it has an aut. pres. using $<_{k-llex}$.

Illustration

\[
\begin{array}{ccc}
\# & 0 & 1 \\
\varepsilon & 0 & 1 \\
\varepsilon & \varphi_0 & \varphi_1
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 11 \\
\varphi_0 & \varphi_1 & \varphi_0 & \varphi_1
\end{array}
\]
Theorem

For all $k$, an $\omega$-word is $k$-morphic iff it has an aut. pres. using $<_k$-lex.

Illustration

\begin{align*}
\# & \quad 0 & \quad \# & \quad 1 & \quad \# \\
\varepsilon & \quad 0 & \quad 1 & \quad 0 & \quad 1 \\
\varepsilon & \quad \varphi_0 & \quad \varphi_1 \\
\end{align*}

\begin{align*}
\# & \quad 0 \quad 0 \quad \# \quad 0 \quad 1 \quad \# \quad 1 \quad 1 \quad \# \\
0 & \quad 0 & \quad 10 & \quad 11 & \quad 00 & \quad 10 & \quad 11 & \quad 00 & \quad 10 & \quad 11 \\
\varphi_0 & \quad \varphi_0 \varphi_1 & \quad \varphi_1 \varphi_0 & \quad \varphi_1 \varphi_1 \\
\end{align*}

Notation

For each $k$, $\mathcal{W}_k$ is the class of $k$-morphic, or $k$-lex, words.
Deciding the MSO theory of $k$-morphic words

$k + 1$-morphic words come with a “built in” depth $k$ factorization represented by $(=0, \ldots, =k)$

**Contraction Lemma** For all $w \in \mathcal{W}_{k+1}$ with “built in” $=k$ and for all $\psi$ we have $w^{\psi}_{f=\mathcal{W}_{k+1}} \in \mathcal{W}_k$ effectively.
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\[
\begin{align*}
\gamma &= [[\#]] \\
\varphi &= [\varphi_0 \varphi_1] \text{ with} \\
\varphi_i : &\begin{cases}
0 &\mapsto 0 \\
1 &\mapsto 1 \\
\# &\mapsto i\# \\
\end{cases} \\
\psi(x) &= |x|_1 \mod 2
\end{align*}
\]
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1
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\tau_{0} \\
\tau_{1}
\end{array} \\
&\begin{array}{c}
0 \\
1
\end{array} \mapsto \begin{array}{c}
0 \\
1
\end{array}
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$k + 1$-morphic words come with a “built in” depth $k$ factorization represented by $(\equiv_0, \ldots, \equiv_k)$

**Contraction Lemma** For all $w \in \mathcal{W}_{k+1}$ with “built in” $\equiv_k$ and for all $\psi$ we have $w^{\psi}_{f_k} \in \mathcal{W}_k$ effectively.

\[ \gamma = [0] \]
\[ \tau : \begin{array}{c|cc} 0 & \tau_0 & \tau_1 \\ \hline 1 & 0 & 1 \\ \end{array} \]

\[ \begin{array}{cccccccc} 0 & 0 & \# & 0 & 1 & \# & 1 & 0 & \# & 1 & 1 & \# \\ \hline \tau_0 \tau_0 & \tau_0 \tau_1 & \tau_1 \tau_0 & \tau_1 \tau_1 \end{array} \]

**Theorem**

*For all $k$, the MSO-theory of every $k$-morphic word is decidable.*
Main Results

Theorem (Main Theorem)

Given \( w \in \mathcal{W}_k \) and \( \varphi(\vec{x}) \in \text{MSO} \) having only first-order variables \( \vec{x} \) free, we can compute an automaton recognizing the relation defined by \( \varphi \) in \( \mathcal{W}_w \).
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Corollaries

- Each $\mathcal{W}_k$ is closed under MSO-definable recolorings.
- If a structure is MSO-interpretable in a $k$-lexicographic word by formulas $\varphi(\vec{x})$ as in the theorem, then it is automatic.
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Corollaries

- Each \( \mathcal{W}_k \) is closed under MSO-definable recolorings.
- If a structure is MSO-interpretable in a \( k \)-lexicographic word by formulas \( \varphi(\vec{x}) \) as in the theorem, then it is automatic.

For each \( k \) consider \( w_k \in \{0, 1, \#\}^\omega \) obtained by concatenating all finite binary words in the \( k \)-lexicographic ordering and separated by hash marks.

Theorem (Characterization)

Let \( w \in \Sigma^\omega \). Then \( w \in \mathcal{W}_k \iff W_w \leq^I W_{w_k} \) for some interpretation \( I = (\varphi_D(x), x < y, \{\varphi_a(x)\}_{a \in \Sigma}) \) such that \( \models \forall x (\varphi_D(x) \rightarrow P_\#(x)) \).
Hierarchy theorem

Clearly, $\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$. 
Hierarchy theorem

Clearly, $\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$.

Consider the following stuttering words defined for each $k$ as

\[
\begin{align*}
    s_0 &= a^\omega \\
    s_1 &= abaaba^4 ba^8 ba^{16} b \ldots \\
    s_2 &= abcaabaabc(a^4 b)^4 c(a^8 b)^8 c \ldots \\
    s_3 &= abcd((a^2 b)^2 c)^2 d((a^4 b)^4 c)^4 d((a^8 b)^8 c)^8 d \ldots \\
    & \vdots \\
    s_k &= \prod_{n=0}^{\infty} (\cdots (((a_0^{2^n}) a_1)^{2^n}) \cdots )^{2^n} a_k \\
    & \vdots
\end{align*}
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 & \vdots \\
 s_k &= \prod_{n=0}^{\infty}(\cdots (((a_0^{2n})a_1)^{2n}) \cdots )^{2n}a_k \\
 & \vdots
\end{align*}
\]

**Theorem (Hierarchy Theorem)**

*For each $k \in \mathbb{N}$ we have $s_{k+1} \in \mathcal{W}_{k+1} \setminus \mathcal{W}_k$.***
Future work and Questions

To do

▶ Locate $\mathcal{W}_k$ in the pushdown hierarchy...
or generate them from simply-typed schemes.
(Cf. tutorial of Didier Caucal on Friday)

▶ Extend results to other (all?) automatic presentations of $(\mathbb{N}, <)$.

** Is isomorphism of $k$-lexicographic words decidable?

** Let $k > k'$. Is it decidable whether a $k$-morphic word is $k'$-lexicographic?
   In particular, is eventual periodicity of $k$-morphic words decidable?
(Cf. same problems for $\omega$-words generated by HD0L systems, i.e. $k = 1$)
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THANK YOU!