Fixed point equations in $\omega$-continuous semirings

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Work in progress with
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Motivation

Interprocedural program analysis: The science of gaining qualitative and quantitative information about the executions of a procedural program without executing it.

Thesis I: $\omega$-continuous semirings are the right foundation

- Program $\longrightarrow$ system of fixed point equations on an abstract semiring
- Analysis problem $\longrightarrow$ concrete semiring
- Algorithmic solution $\longrightarrow$ equation solver

Thesis II: efficient algorithms are accelerations of Knaster-Tarski’s iteration (compute $f(0), f^2(0), f^3(0)\ldots$)

Result: two apparently unconnected accelerations from the literature coincide
ω-continuous semirings and fixed point equations

Algebra \((C, +, \cdot, 0, 1)\) satisfying the following axioms

– \((C, +, 0)\) is a commutative monoid
– \(\cdot\) distributes over \(+\)
– \((C, \cdot, 1)\) is a monoid
– “Infinite sums” exist

System of equations \(\mathbf{x} = f(\mathbf{x})\) where

– \(\mathbf{x} = (x_1, \ldots, x_n)\) vector of variables,
– \(f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))\) vector of terms over \(C \cup \{x_1, \ldots, x_n\}\)

Example:

\[
\begin{align*}
  x_1 &= a \cdot x_1 \cdot x_2 + b \cdot x_2 \\
  x_2 &= b \cdot x_1 \cdot x_1 + c
\end{align*}
\]

Problem: compute the least fixed point \(\mathbf{x}_{\text{lfp}}\)
Connection to program analysis

Probabilistic context-free process:

\[ X \xrightarrow{a, p} XX \]
\[ X \xrightarrow{b, (1 - p)} \varepsilon \]

Abstract fixed point equation:

\[ X = r_1 \cdot X \cdot X + r_2 \]
Problem 1: set of executions?
Languages over \( \{a, b\} \), union and concatenation, \( r_1 = a \), \( r_2 = b \) + is idempotent

Problem 2: number of \( a \)'s and \( b \)'s in executions?
\( (\mathbb{N} \cup \{\omega\})^2 \), union and vector sum, \( r_1 = (0, 1) \), \( r_2 = (1, 0) \) + is idempotent, \( \cdot \) is commutative

Problem 3: probability of termination?
\( \mathbb{R}^+ \cup \{\infty\} \), sum and product, \( r_1 = p \), \( r_2 = 1 - p \) \( \cdot \) is commutative

Problem 4: probability of the most likely execution?
\( \mathbb{R}^+ \cup \{\infty\} \), \( \text{max} \) and product, \( r_1 = p \), \( r_2 = 1 - p \).
+ is idempotent, \( \cdot \) is commutative
Iterative solution

Theorem [Knaster-Tarski]: $\vec{x}_{lfp}$ is the supremum of $\{\vec{x}_i\}_{i \geq 0}$, where

\begin{align*}
\vec{x}_0 &= \vec{0} \\
\vec{x}_{i+1} &= f(\vec{x}_i)
\end{align*}

Language case: does not terminate if $\vec{x}_{lfp}$ is an infinite language.

Probabilistic case: convergence can be exponentially slow [EY05].

In $x = 1/2x^2 + 1/2$ we have $x_{\text{lfp}} = 1$, but

$$x_{2k} \leq 1 - \frac{1}{2^k + 1}$$
Improving the efficiency

Look for (formal) accelerations based on the Kleene star operator

\[ a^* = \sum_{i \geq 0} a^i \]

Whether the acceleration speeds up the computation or not depends on techniques for computing \( a^* \).
Acceleration I: Newton’s method

Real domain $\mathbb{R}^n$

Newton’s method for approximating a zero of a real function in one variable:

\[ x_0 = \text{a value “close enough” to the zero} \]
\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

Can be transformed into an iteration method for approximating a fixed point (Etessami and Yannakakis, STACS’05)

To approximate a fixed point of $f(x)$, approximate a zero of $f(x) - x$:

\[ x_0 = 0 \]
\[ x_{i+1} = x_i + \frac{f(x_i) - x_i}{1 - f'(x_i)} \]
Multivariate version:

$$\begin{align*}
\vec{x}_0 &= \vec{0} \\
\vec{x}_{i+1} &= \vec{x}_i + \left( I - \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^{-1} \cdot f(\vec{x}_i)
\end{align*}$$

Theorem [EY05]: If the system of equations is “strongly connected”, Newton’s method converges to $x_{lfp}$.

Otherwise, the method can be applied “SCC-wise”
Arbitrary semiring where $+$ is idempotent and $\cdot$ is commutative.

Acceleration due to Hopkins and Kozen, LICS ’99

Define the formal partial differential operator $\frac{\partial}{\partial x_i}$ by

$$\frac{\partial a}{\partial x_i} = 0 \quad \frac{\partial}{\partial x_i} (g + h) = \frac{\partial g}{\partial x_i} + \frac{\partial h}{\partial x_i}$$

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij} \quad \frac{\partial}{\partial x_i} (g \cdot h) = \frac{\partial g}{\partial x_i} \cdot h + g \cdot \frac{\partial h}{\partial x_i}$$
Theorem [HK99]: If $+$ is idempotent and $\cdot$ is commutative, then $\vec{x}_{\text{lfp}}$ is the supremum of $\{\vec{x}_i\}_{i \geq 0}$ defined by

$$
\vec{x}_0 = \vec{0} \\
\vec{x}_{i+1} = \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* \cdot f(\vec{x}_i)
$$

where

$$
\frac{\partial f}{\partial \vec{x}} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
$$

(Jacobi matrix)
Unified acceleration scheme

Theorem: If $\cdot$ is commutative then $\vec{x}_{lfp}$ is the supremum of $\{\vec{x}_i\}_{i \geq 0}$, where

$$
\vec{x}_0 = \vec{0}
$$

$$
\vec{x}_{i+1} = \vec{x}_i + \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* \cdot \vec{\delta}_i
$$

and $\vec{\delta}_i$ is any vector satisfying $\vec{x}_i + \vec{\delta}_i = f(\vec{x}_i)$

Theorem (informally): The accelerations I and II are instances of this one.

Corollary (more informally): [HK99] was discovered by Newton 300 years ago.
Since $x_i \leq f(\vec{x}_i)$ we have $x_i + f(\vec{x}_i) = f(\vec{x}_i)$ and so $\vec{\delta}_i := f(\vec{x}_i)$.

We have

$$\vec{x}_0 = 0$$
$$\vec{x}_{i+1} = \vec{x}_i + \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* \cdot \vec{\delta}_i$$

Since $\vec{x}_i \leq \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* f(\vec{x}_i)$ we get

$$\vec{x}_0 = 0$$
$$\vec{x}_{i+1} = \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* f(\vec{x}_i)$$
Newton’s method [EY05]

Consider the univariate case: \( \frac{\partial f}{\partial x} \implies f' \)

Since \( x_i + \delta_i = f(x_i) \) we have \( \delta_i := f(x_i) - x_i \)

If \( f'(x_i) < 1 \) then

\[
f'(x_i)^* = 1 + f'(x_i) + f'(x_i)^2 + \ldots = \frac{1}{1 - f'(x_i)}
\]

We get

\[
\begin{align*}
x_0 &= 0 \\
x_{i+1} &= x_i + f'(x_i)^* \cdot \delta_i
\end{align*}
\]
In the multivariate case

\[ \vec{x}_0 = \vec{0} \]
\[ \vec{x}_{i+1} = \vec{x}_i + \left( I - \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^{-1} \cdot (f(\vec{x}_i) - \vec{x}_i) \]

The generalization provides a new algorithm to approximate the fixed point:

- compute \( \left( \frac{\partial f}{\partial \vec{x}}(\vec{x}_i) \right)^* \) by divide and conquer
- no need to decompose in SCCs
Questions

Is there a generic acceleration scheme for arbitrary $\omega$-continuous semirings (not necessarily commutative) ?

How many iterations does [HK99] need?
$O(3^n)$ in [HK99], we think to have proved $O(n)$.

How fast does Newton converge in the real domain ?

Fundamental question in the theory of PPDs

Surprisingly, nobody seems to know!!