
PART I - $N$-Koszul algebras


Why higher $N$'s?

1. There are some relevant examples coming from
- noncommutative projective geometry: cubic Artin-Schelter regular algebras (1987) of global dimension 3, such as

\[ A = \frac{\mathbb{C}\langle x, y \rangle}{(ay^2x + byxy + axy^2 + cx^3, x \leftrightarrow y)} \]


\[ A = \frac{\mathbb{C}\langle \nabla_0, \ldots, \nabla_s \rangle}{(\sum_{\lambda \mu} g^{\lambda \mu} [\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]])} \]

where \( g^{\lambda \mu} \) are entries of an invertible symmetric real matrix.


\[ A = \frac{\mathbb{C}\langle x_1, \ldots, x_n \rangle}{(\sum_{\sigma} \text{sgn}(\sigma) x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(N)}})} \]

for \( 2 \leq N \leq n \).
2. Poincaré duality in Hochschild (co)homology (N. Marconnet and R.B. 2003). Using Van den Bergh’s duality theorem, we proved that if $A$ is $N$-Koszul and AS-Gorenstein of global dimension $d$, then

$$HH^i(A, M) \cong HH_{d-i}(A, \varepsilon^{d+1}\phi M).$$

Moreover (R. Taillefer and R.B.), $A$ is Calabi-Yau (of dimension $d$) if and only if the automorphism $\varepsilon^{d+1}\phi$ of $A$ is the identity automorphism.

Some applications of the latter result: the first two examples are Calabi-Yau. For $N \geq 3$, the third one is Calabi-Yau if and only if $N$ divides $n - 1$ and $n$ is odd.
3. $N$-Koszul algebras and quantum Mac-Mahon Master Theorem (qMMT)

Let $A$ be an $N$-Koszul superalgebra. Extending Manin’s quadratic approach of quantum groups (Montréal 1988) to higher $N$, the superbialgebra $B = \text{end}(A)$ “of matrices over $A$” is still defined (M. Dubois-Violette, M. Wambst and R.B. 2003).

Then, Phùng Hồ Hai, B.Kriegk, and M. Lorenz recently proved the following “$N$-Koszul MMT”:

$$\left( \sum_{\ell \geq 0} \chi^s(A_\ell)t^\ell \right) \left( \sum_{m \geq 0} (-1)^m \chi^s(A^{\dagger\ast}_{\nu_N}(m))t^{\nu_N(m)} \right) = 1,$$

where $\chi^s(M) \in B$ denotes the supercharacter of any finite-dimensional $B$-comodule $M$.

Specialising $N = 2$ and $A = \text{multiparameter quantum space}$, $\text{qMMT}$ (by Garoufalidis-Lè-Zeilberger and Konvalinka-Pak) is recovered.
PART II - Basic observation: Bocklandt’s algebras $A$ are $N$-Koszul

$A$ is a graded Calabi-Yau algebra of dimension 3. As proved by Bocklandt, $A$ is defined by a homogeneous potential $W_{N+1}$ of degree $N+1$:

$$A = A(Q, W_{N+1}) := \mathbb{C}Q/(\partial a W_{N+1}; \ a \in Q_1).$$

Assume $\#Q_0 = 1$ and $Q_1 = \{x_1, \ldots, x_n\}$. Then we have the following projective resolution of $A$ as an $A$-bimodule:

$$0 \to A^e \xrightarrow{\delta_3} (A^e)^n \xrightarrow{\delta_2} (A^e)^n \xrightarrow{\delta_1} A^e \xrightarrow{\mu} A \to 0,$$

where

$$\delta_1((a_j \otimes b_j)_{1 \leq j \leq n}) = \sum_{1 \leq j \leq n} a_j x_j \otimes b_j - a_j \otimes x_j b_j$$

$$\delta_2((a_i \otimes b_i)_{1 \leq i \leq n}) = \left( \sum_{1 \leq i \leq n} a_i \frac{\partial^2 W_{N+1}}{\partial x_i \partial x_j} b_i \right)_{1 \leq j \leq n}$$

$$\delta_3(a \otimes b) = (a \otimes x_i b - ax_i \otimes b)_{1 \leq i \leq n}.$$
PART III - PBW deformations of a graded algebra $A = \mathbb{C}Q/(R)$, where $R$ is any $\mathbb{C}Q_0$-subbimodule of $\mathbb{C}Q_N$.

Let $P$ be a $\mathbb{C}Q_0$-subbimodule of $\mathbb{C}Q_{\leq N}$ such that $\pi(P) = R$, where $\pi : \mathbb{C}Q_{\leq N} \rightarrow \mathbb{C}Q_N$ is the natural projection.

Set $U = \mathbb{C}Q/(P)$. It is a filtered algebra, and there is a natural graded algebra morphism $p : A \rightarrow \text{gr}(U)$.

**Definition.** $U$ is called a PBW deformation of $A$ if $p$ is an isomorphism, that is, if any monomial basis of $A$ is also a basis of $U$.

**Theorem** (V. Ginzburg and R.B. 2006) Assume that $A$ is $N$-Koszul. Then $U$ is a PBW deformation of $A$ if and only if we have

$$P \cap \mathbb{C}Q_{\leq N-1} = 0,$$

$$\left(P \otimes_{\mathbb{C}Q_0} V + V \otimes_{\mathbb{C}Q_0} P\right) \cap \mathbb{C}Q_{\leq N} \subseteq P.$$
Condition (1) is a non-triviality condition.

Condition (2) is an $N$-generalisation of the Jacobi identity: for $N = 2$, assume $\#Q_0 = 1$ and $Q_1 = \{x_1, \ldots, x_n\}$, and choose an alternating bilinear map

$$[[,] : \mathbb{C}Q_1 \times \mathbb{C}Q_1 \rightarrow \mathbb{C}Q_1.$$ 

Let $P$ be spanned over $\mathbb{C}$ by the elements

$$x \otimes y - y \otimes x - [x, y], \ x \in Q_1, \ y \in Q_1.$$ 

Then (1) holds, and (2) is equivalent to the Jacobi identity of the bracket $[[,]]$. In this case, $U$ is the enveloping algebra of the Lie algebra $(\mathbb{C}Q_1,[[,]])$. 

PART IV - PBW deformations of a Bocklandt algebra $A = A(Q, W_{N+1})$. Then

$$R = \text{span}_{\mathbb{C}Q_0}\{\partial_a W_{N+1}; \ a \in Q_1\}.$$  

We apply the previous theorem to get the following.

**Theorem** (R. Taillefer and R.B. 2006) Let $W'$ be a potential of degree $< N + 1$. Then

$$U = A(Q, W = W_{N+1} + W')$$

is a PBW deformation of $A$, and $U$ is Calabi-Yau of dimension 3.

Let us discuss only the first point. We have

$$P = \text{span}_{\mathbb{C}Q_0}\{\partial_a W; \ a \in Q_1\},$$

$$P \cap \mathbb{C}Q_{\leq N-1} = \{\sum_a \lambda_a \partial_a W; \sum_a \lambda_a \partial_a W_{N+1} = 0\}.$$  

Since $A$ is Calabi-Yau of dimension 3, $\#Q_1 = \dim R$, thus the $\partial_a W_{N+1}$'s are linearly independent, and condition (1) holds.
PART V - A noncommutative Euler identity

Euler identity: for a (commutative) polynomial \( W \) in variables \( x_1, \ldots, x_n \),

\[
\sum_{1 \leq i \leq n} x_i \frac{\partial W}{\partial x_i} = c(W),
\]

where \( c \) is the cyclic sum of each monomial.

Analogue in circular calculus (one vertex case): for a potential \( W \) in \( x_1, \ldots, x_n \),

\[
\sum_{1 \leq i \leq n} x_i \partial_{x_i} W = c(W) = \sum_{1 \leq i \leq n} \partial_{x_i} W x_i.
\]

For example, if \( W = xyz - yxz \), then

\[
\partial_x W = yz - zy, \quad \partial_y W = zx - xz, \quad \partial_z W = xy - yx
\]

and \((x \partial_x + y \partial_y + z \partial_z)(W) = c(W)\) is the totally antisymmetric tensor in \( x, y, z \).

General case: for each vertex \( e \) of a quiver \( Q \), we have

\[
\sum_{a \in eQ_1} a \partial_a W = e \cdot c(W) = \sum_{b \in Q_1 e} \partial_b W b.
\]
PART VI - The space $\mathcal{W}$ of volume forms of $A = A(Q, W_{N+1})$

It is $\mathcal{W} = (R \otimes_{CQ_0} CQ_1) \cap (CQ_1 \otimes_{CQ_0} R)$. For each vertex $e$, $e \cdot c(W_{N+1})$ belongs to $\mathcal{W}$.

Since $A$ is Calabi-Yau of dimension 3, $\#Q_0 = \dim \mathcal{W}$, thus $(e \cdot c(W_{N+1}))_{e \in Q_0}$ is a basis of $\mathcal{W}$.

Recall that $U = A(Q, W_{N+1} + W')$.

Define $\varphi : R \to CQ_{\leq N-1}$ by $\varphi(\partial a W_{N+1}) = -\partial a W'$, so that relations of $U$ are

$$\partial a W_{N+1} = \varphi(\partial a W_{N+1}), \quad a \in Q_1.$$ 

Clearly $\varphi \otimes 1 := \varphi \otimes_{CQ_0} \text{id}_{CQ_1}$ and $1 \otimes \varphi := \text{id}_{CQ_1} \otimes_{CQ_0} \varphi$ are defined on $\mathcal{W}$.

We know (V. Ginzburg and R.B.) that condition (2) is equivalent to

$$(\varphi \otimes 1 - 1 \otimes \varphi)(\mathcal{W}) \subseteq P.$$
Using circular Euler identity, we calculate for each vertex \( e \),

\[
(\varphi \otimes 1 - 1 \otimes \varphi)(e.c(W_{N+1}))
\]

\[
= \sum_{a \in eQ_1} a \partial_a W' - \sum_{b \in Q_1 e} \partial_b W' b,
\]

which is zero, again from circular Euler identity.

Thus we get a condition stronger than (2):

\[
(P \otimes_{\mathbb{C}Q_0} V + V \otimes_{\mathbb{C}Q_0} P) \cap \mathbb{C}Q_{\leq N} = 0. \tag{3}
\]

The converse statement is true:

**Theorem** (R. Taillefer and R.B. 2006) If \( U = \mathbb{C}Q/(P) \) is a PBW deformation of a Bocklandt algebra \( A = A(Q, W_{N+1}) \) such that condition (3) holds, then there exists a unique potential \( W' \) without constant term such that

\[
U = A(Q, W_{N+1} + W').
\]