Cambridge 2007

QUANTUM FIELDS AND SCALE INVARIANCE ON STAR GRAPHS

B. Bellazzini, M. Burrello, M. M. (Pisa)
P. Sorba, (Annecy)

Plan

I. Introduction.

1. Free scalar fields on star graphs.
2. Symmetries and Kirchhoff's rule.
3. Reflection-transmission (RT) algebra $\mathcal{A}$.
4. Fock and Gibbs representation of $\mathcal{A}$.
5. Scale invariance, critical points and flows.
6. Casimir energy and Casimir force.

II. Vertex operators on star graphs.

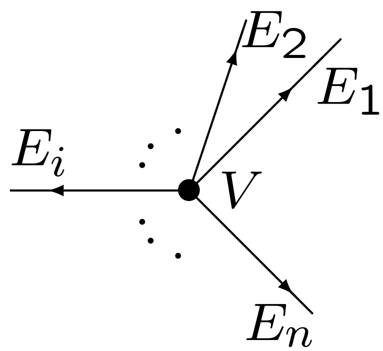
1. Basic definition and correlation functions.
2. Bosonization of fermions.
3. Conductance.

III. Nontrivial bulk interactions.

1. The massless Thirring model.
2. The nonlinear Schrödinger model.

IV. Conclusions.

I. Introduction



A star graph Γ with n edges.

Each point P in Γ is parametrized by:

$x \in \mathbb{R}_+$ - distance of P from V;

$i = 1, ..., n$ - index of the edge.

$B \equiv \Gamma \setminus V$ - bulk of Γ.

1. Scalar field on Γ.

$$\varphi(t,x,i)\ :\ \mathcal{D} \to \mathcal{D}\,, \qquad \overline{\mathcal{D}} = \mathcal{H}\,,$$

satisfying:

- equation of motion:

$$\left(\partial_t^2 - \partial_x^2\right)\varphi(t,x,i) = 0\,, \qquad x > 0\,,$$

- initial condition (equal-time CCR):

$$[\varphi(t,x_1,i_1)\,,\,\varphi(t,x_2,i_2)] =$$

$$[(\partial_t\varphi)(t,x_1,i_1)\,,\,(\partial_t\varphi)(t,x_2,i_2)] = 0\,,$$

$$[(\partial_t\varphi)(t,x_1,i_1)\,,\,\varphi(t,x_2,i_2)] = -\mathrm{i}\delta_{i_1}^{i_2}\,\delta(x_1-x_2)\,.$$

- boundary condition: $\forall\, t \in \mathbb{R}$

$$\sum_{j=1}^{n}\left[A_i^j\,\varphi(t,0,j) + B_i^j(\partial_x\varphi)(t,0,j)\right] = 0\,,$$

where A, B are $n \times n$ complex matrices.

The theory of ∂_x^2 on Γ (Kostrykin, Schrader, Harmer,...) implies unitary (dissipationless) time evolution of φ iff

$$A\,B^* - B\,A^* = 0\,, \qquad \text{rank}\,(A, B) = n\,.$$

Equivalent pairs:

$$\{A, B\} \sim \{CA, CB\}\,, \qquad C - \text{invertible}\,.$$

Supplementary conditions on φ:

(i) Hermiticity: $\varphi^*(t, x, i) = \varphi(t, x, i)$

(ii) Time-reversal invariance:

$$T\varphi(t, x, i)T^{-1} = \varphi(-t, x, i)\,, \quad T - \text{antilinear}\,.$$

Up to a C-factor A and B are real and satisfy

$$A\,B^t - B\,A^t = 0\,, \qquad \text{rank}\,(A, B) = n\,.$$

The solution in algebraic terms:

$$\varphi(t,x,i) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi\sqrt{2|k|}}$$
$$\left[a^{*i}(k)\mathrm{e}^{\mathrm{i}(|k|t-kx)} + a_i(k)\mathrm{e}^{-\mathrm{i}(|k|t-kx)}\right],$$

$\{a_i(k),\, a^{*i}(k)\ :\ k \in \mathbb{R}\}$ generate an associative algebra $\mathcal{A}$ with identity element $\mathbf{1}$ and satisfy the commutation relations

$$\begin{aligned} a_{i_1}(k_1)\, a_{i_2}(k_2) - a_{i_2}(k_2)\, a_{i_1}(k_1) &= 0\,, \\ a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a^{*i_1}(k_1) &= 0\,, \\ a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a_{i_1}(k_1) &= \end{aligned}$$
$$2\pi[\delta_{i_1}^{i_2}\delta(k_1-k_2) + S_{i_1}^{i_2}(k_1)\delta(k_1+k_2)]\mathbf{1}\,,$$

and the constraints

$$a_i(k) = S_i^j(k)a_j(-k)\,, \quad a^{*i}(k) = a^{*j}(-k)S_j^i(-k)\,,$$

where $S(k)$ is the S-matrix characterizing the defect.

$\mathcal{A}$ is a special case of the reflection-transmission algebras, introduced (Sorba, Ragoucy, Caudrelier, M. M,...) for dealing with impurities in QFT.

In our case (Kostrykin, Schrader, Harmer,...):

$$S(k) = -(A + \mathrm{i}kB)^{-1}(A - \mathrm{i}kB) \,.$$

Some useful properties of $S(k)$:

- Unitarity:

$$S(k)^* = S(k)^{-1} \,;$$

- Hermitian analyticity:

$$S(k)^* = S(-k) \,;$$

- Invariance under time reversal:

$$S(k)^t = S(k) \,.$$

As a consequence, one has

$$S(k)\, S(-k) = \mathbb{I}_n \,,$$

which ensures the consistency of the constraints:

$$a_i(k) = S_i^j(k) a_j(-k) \,, \quad a^{*i}(k) = a^{*j}(-k) S_j^i(-k) \,.$$

Symmetries and Kirchhoff's rule

Symmetries in QFT are associated with conserved currents

$$\partial_t j_t(t,x,i) - \partial_x j_x(t,x,i) = 0\,.$$

The concept of symmetry on Γ needs special attention.

In analogy with electromagnetism, the total charge

$$Q = \sum_{i=1}^{n} \int_0^\infty \mathrm{d}x\, j_t(t,x,i)$$

is conserved iff the relative Kirchhoff's rule

$$\sum_{i=1}^{n} j_x(t,0,i) = 0$$

holds in the vertex V of Γ.

Examples:

(a) Energy conservation

$$\theta_{tt}(t,x,i) = \frac{1}{2} : \Big[(\partial_t \varphi)(\partial_t \varphi) - \varphi(\partial_x^2 \varphi)\Big] : (t,x,i)\,,$$

$$\theta_{tx}(t,x,i) = \frac{1}{2} : [(\partial_t \varphi)(\partial_x \varphi) - \varphi(\partial_t \partial_x \varphi)] : (t,x,i)\,,$$

where $: \cdots :$ denotes the normal product in the algebra $\mathcal{A}$. The associated Kirchhoff rule

$$\sum_{i=1}^{n} \theta_{tx}(t,0,i) = 0$$

is satisfied by construction (Kostrykin,...).

(b) $U(1)$-symmetry of the vertex operators

$$j_t(t,x,i) = \partial_t \varphi(t,x,i)\,, \qquad j_x(t,x,i) = \partial_x \varphi(t,x,i)\,.$$

Kirchhoff's rule for this symmetry implies the following additional constraint on A:

$$\mathbf{v} \equiv (1,1,...,1) \in \mathrm{Ker}A$$

Reflection-transmission (RT) algebra $\mathcal{A}$

$$
\begin{aligned}
a_{i_1}(k_1)\, a_{i_2}(k_2) - a_{i_2}(k_2)\, a_{i_1}(k_1) &= 0\,, \\
a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a^{*i_1}(k_1) &= 0\,, \\
a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a_{i_1}(k_1) &=
\end{aligned}
$$

$$
2\pi[\delta_{i_1}^{i_2}\delta(k_1 - k_2) + S_{i_1}^{i_2}(k_1)\delta(k_1 + k_2)]\mathbf{1}\,,
$$

$\mathcal{A}$ translates in algebraic terms the boundary value problem at hand.

$\mathcal{A}$ defines convenient coordinates in the field space, which simplify the computation of the correlation functions.

The Hamiltonian for instance has the simple form

$$
H = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \omega(k) a^{*i}(k) a_i(k)\,,
$$

where $\{a_i(k), a^{*i}(k)\}$ keep track of $S(k)$.

We will adopt two representations of $\mathcal{A}$:

Fock representation- cyclic state Ω_F:

$$(A_1\Omega_F\,,\,A_2\Omega_F)\equiv\langle A_1^*A_2\rangle_F$$

$$\begin{aligned}\langle a_i(p)a^{*j}(q)\rangle_F &= 2\pi\left[\delta_i^j\,\delta(p-q)+S_i^j(p)\,\delta(p+q)\right]\,,\\ \langle a^{*i}(p)a_j(q)\rangle_F &= 0\,.\end{aligned}$$

Gibbs representation - cyclic state Ω_G:

$$\langle a_i(p)a^{*j}(q)\rangle_\beta = \frac{1}{1-\mathrm{e}^{-\beta|p|}}2\pi\left[\delta_i^j\delta(p-q)+S_i^j(p)\delta(p+q)\right]\,,$$

$$\langle a^{*i}(p)a_j(q)\rangle_\beta = \frac{\mathrm{e}^{-\beta|p|}}{1-\mathrm{e}^{-\beta|p|}}2\pi\left[\delta_j^i\delta(p-q)+S_j^i(-p)\delta(p+q)\right]\,,$$

$\beta=\frac{1}{T}$ - inverse temperature;

All other correlators can be expressed in terms of the above ones.

The physical observable in this context is $S(k)$.

Problem: Express A and B in terms of $S(k)$. Actually, it is enough (Kostrykin, Schrader, Harmer,...) to know

$$S_0 = S(k_0)\,.$$

Recalling that

$$S_0^* = S_0^{-1}\,, \qquad S_0^t = S_0\,,$$

one has

$$A = \frac{1}{2}C(\mathbb{I}_n - S_0)\,, \qquad B = -\frac{\mathrm{i}}{2k_0}C(\mathbb{I}_n + S_0)\,,$$

where C can be fixed in such a way that A and B are real. Finally,

$$S(k) = \frac{(k-k_0)\mathbb{I}_n + (k+k_0)S_0}{(k+k_0)\mathbb{I}_n + (k-k_0)S_0}\,,$$

5. Scale invariance and critical points.

Invariance under scale transformations:

$$x \longmapsto \rho x\,, \qquad t \longmapsto \rho t\,, \qquad \rho > 0\ .$$

Scale invariant (critical) points: simple universal properties.

The scale invariant S-matrices are generated by S_0 satisfying (Harmer):

$$S_0^* = S_0^{-1}\,, \text{(unitarity)} \qquad S_0^t = S_0\,, \text{(time} - \text{rev.)}$$

$$S_0\mathbf{v} = \mathbf{v}\,, (U(1) - \text{Kirchhoff}) \qquad S_0^* = S_0\,, \text{(scale} - \text{inv.)}$$

The scale invariant S-matrices are k-independent.

Families of scale invariant points; $p = \text{rank}\, A$:

$p = 0 \Rightarrow S_N = \mathbb{I}$ - (Neumann);

$p = n \Rightarrow S_D = -\mathbb{I}$ - (Dirichlet);

$0 < p < n \Rightarrow p(n-p-1)$-parameter family;

Classification of the scale invariant S-matrices for $n = 3$:

$$S_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(Neumann)}$$

and

$$S_{K-F} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \text{(Kane, Fisher, ...)}$$

are symmetric under edge permutations.

The S-matrices ($\alpha \in \mathbb{R}$)

$$S_\alpha = \frac{1}{1+\alpha+\alpha^2} \begin{pmatrix} \alpha+1 & -\alpha & \alpha(\alpha+1) \\ -\alpha & \alpha(\alpha+1) & \alpha+1 \\ \alpha(\alpha+1) & \alpha+1 & -\alpha \end{pmatrix}$$

are not symmetric and appear to be new in the physical context.

6. Casimir energy. (Fulling, Harrison,...)

$$\theta_{tt}(t,x,i) = \frac{1}{2}\left[(\partial_t\varphi)(\partial_t\varphi) - \varphi(\partial_x^2\varphi)\right](t,x,i)\,,$$

The point-splitting procedure gives

$$\langle\theta_{tt}(t,x,i)\rangle_\beta - \langle\theta(t,x)\rangle_\infty^{\text{line}} =$$

$$\varepsilon_{\text{S-B}}(\beta) + \mathcal{E}_{\text{C}}(x,i) + \mathcal{E}(x,i,\beta)\,,$$

with:

- Stefan-Boltzmann contribution

$$\varepsilon_{\text{S-B}}(\beta) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,|k|\,\frac{e^{-\beta|k|}}{1-e^{-\beta|k|}} = \frac{\pi}{6\beta^2} \sim T^2\,,$$

- Casimir energy density

$$\mathcal{E}_{\text{C}}(x,i) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,|k|\,S_i^i(k)e^{2\mathrm{i}kx}$$

- Correction to $\varepsilon_{\text{S-B}}(\beta)$ and/or $\mathcal{E}_{\text{C}}(x,i)$

$$\mathcal{E}(x,i,\beta) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,|k|\,\frac{e^{-\beta|k|}}{1-e^{-\beta|k|}}\,S_i^i(k)e^{2\mathrm{i}kx}\,.$$

The scale invariant case.

The integration over k is trivial because S is k-independent. One finds:

$$\mathcal{E}(x,i,\beta) = \frac{\pi S_i^i}{2\beta^2 \sinh^2\left(2\pi\frac{x}{\beta}\right)} - \frac{S_i^i}{8\pi x^2},$$

$$\mathcal{E}_{\mathrm{C}}(x,i) = -\frac{S_i^i}{8\pi x^2}.$$

Stability of critical points:

$$\mathcal{E}_{\mathrm{C}}(x) = \sum_{i=1}^{n} \mathcal{E}_{\mathrm{C}}(x,i) = -\frac{1}{8\pi x^2}\mathrm{Tr}\, S.$$

The case $n = 3$:

$$S_{K-F} \longrightarrow S_\alpha \longrightarrow S_N$$

II. Vertex algebras on star graphs.

- The dual field $\widetilde{\varphi}$.

$$\partial_t \widetilde{\varphi}(t,x,i) = -\partial_x \varphi(t,x,i)\,,$$

$$\partial_x \widetilde{\varphi}(t,x,i) = -\partial_t \varphi(t,x,i)\,,$$

- Right and left chiral fields:

$$\varphi_{i,R}(t-x) = \varphi(t,x,i) + \widetilde{\varphi}(t,x,i)\,,$$

$$\varphi_{i,L}(t+x) = \varphi(t,x,i) - \widetilde{\varphi}(t,x,i)\,.$$

- Vertex operators: $\zeta = (\sigma, \tau) \in \mathbb{R}^2$

$$V(t,x,i;\zeta) \sim$$

$$\sim : \exp\left\{ \mathrm{i}\sqrt{\pi}\left[\sigma\varphi_{i,R}(t-x) + \tau\varphi_{i,L}(t+x)\right]\right\} : \;.$$

$: \cdots :$ being the normal product in $\mathcal{A}$.

Correlation functions - scale-invariant case:

$$\langle V(t_1, x_1, i_1; \zeta) V^*(t_2, x_2, i_2; \zeta) \rangle_F \sim$$

$$\left[\frac{1}{\mathrm{i}(t_{12} - x_{12}) + \epsilon}\right]^{\sigma^2 \delta_{i_1}^{i_2}} \left[\frac{1}{\mathrm{i}(t_{12} + x_{12}) + \epsilon}\right]^{\tau^2 \delta_{i_1}^{i_2}}$$

$$\left[\frac{1}{\mathrm{i}(t_{12} - \widetilde{x}_{12}) + \epsilon}\right]^{\sigma\tau S_{i_1}^{i_2}} \left[\frac{1}{\mathrm{i}(t_{12} + \widetilde{x}_{12}) + \epsilon}\right]^{\sigma\tau S_{i_1}^{i_2}}$$

with

$$t_{12} = t_1 - t_2\,, \quad x_{12} = x_1 - x_2\,, \quad \widetilde{x}_{12} = x_1 + x_2.$$

Scaling matrix: $(x \longmapsto \rho x,\ t \longmapsto \rho t)$

$$D = (\sigma^2 + \tau^2)\mathbb{I}_n + 2\sigma\tau S\,.$$

Scaling dimensions - eigenvalues of D:

$$d_i = \frac{1}{2}(\sigma + s_i \tau)^2 \geq 0\,,$$

s_i being the eigenvalues of S.

2. Bosonization of fermions. The massless Dirac equation on Γ is

$$(\gamma_t \partial_t - \gamma_x \partial_x)\psi(t,x,i) = 0\,,$$

where

$$\psi(t,x,i) = \begin{pmatrix} \psi_1(t,x,i) \\ \psi_2(t,x,i) \end{pmatrix}\,,$$

$$\gamma_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \gamma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

Solution in terms of vertex operators:

$$\psi_1(t,x,i) \sim V(t,x,i;\zeta = (1,0))\,,$$

$$\psi_2(t,x,i) \sim V(t,x,i;\zeta = (0,1))\,.$$

Conserved current ($\nu = t, x$):

$$j_\nu(t,x,i) = \overline{\psi}(t,x,i)\gamma_\nu \psi(t,x,i) \sim \partial_\nu \varphi(t,x.i)\,.$$

Boundary condition in terms of the current:

$$\sum_{j=1}^{n} A_i^j \int_{+0}^{\infty} \mathrm{d}x\, j_x(t,x,j) = \sum_{j=1}^{n} B_i^j\, j_x(t,0,j)\,.$$

3. Conductance:

Couple the system with an external classical field $A_\nu(t,x,i)$ according to

$$[\gamma_t(\partial_t + \mathrm{i}A_t) - \gamma_x(\partial_x + \mathrm{i}A_x)]\psi = 0$$

and compute the expectation value

$$\langle j_x(t,x,i)\rangle_{A_\nu}\,.$$

Result in the linear response approximation:

$$\langle j_x(t,0,i)\rangle_{A_x} = \frac{1}{2}\sum_{j=1}^{n}\left(\delta_i^j - S_i^j\right)A_x(t,j)\,.$$

Therefore the conductance tensor is

$$G_i^j = G_{\mathsf{line}}\left(\delta_i^j - S_i^j\right)\,,$$

where G_{line} is the conductance of a single wire.

Enhancement of the conductance:

$$G_i^i > G_{\mathsf{line}} \qquad \text{for} \qquad S_i^i < 0\,.$$

General features of the conductance tensor:

- Kirchhoff's rule:

$$\sum_{j=1}^{n} G_i^j = 0\,, \qquad \forall\, i = 1, ..., n\,.$$

- Unitarity bound:

$$0 \le G_i^i \le 2G_{\text{line}}\,.$$

- Sum rule

$$\text{Tr}\, G = 2p\, G_{\text{line}}\,, \qquad p = \text{rank}\ A\,.$$

Relation conductance - Casimir force

$$\mathcal{F}_{\text{C}}(x,i) = \partial_x \mathcal{E}_{\text{C}}(x,i)$$

$$G_i^j = G_{\text{line}} \left(\delta_i^j - S_i^j\right)\,, \qquad \mathcal{E}_{\text{C}}(x,i) = -\frac{S_i^i}{8\pi x^2}\,,$$

enhanced conductance ($S_i^i < 0$) - repulsive force; depressed conductance ($S_i^i > 0$) - attractive force.

The K-F critical point - enhanced conductance:

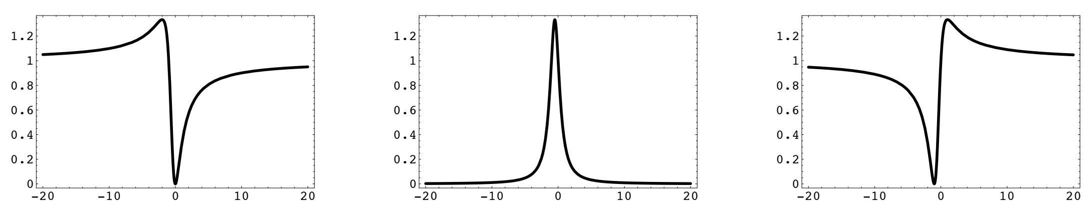
$$G_1^1 = G_2^2 = G_3^3 = \frac{4}{3} G_{\text{line}} \, .$$

Physical explanation (Nayak, Fisher, Ludwig,...)
- Andreev reflection from the vertex.

The α critical points - mixed properties:
The sum rule gives ($p = 1$):

$$G_1^1(\alpha) + G_2^2(\alpha) + G_3^3(\alpha) = 2 G_{\text{line}} \, ,$$

The conductance as a function of α for $G_{\text{line}} = 1$:



$G_1^1(\alpha)$ $\qquad$ $G_2^2(\alpha)$ $\qquad$ $G_3^3(\alpha)$

Domains of enhancement (maxima $= 4G_{\text{line}}/3$);

Domains of depression (minima $= 0$)

A final comment on conductance

- By scale invariance at any critical point

$$G_i^j \in \mathbb{R}\,,$$

implying pure resistance.

- Away from criticality, one gets

$$G_i^j(\omega) = G_{\text{line}}\left(\delta_i^j - S_i^j(\omega)\right)\,,$$

where ω is the frequency of the external field A_ν.

In general $G_i^j(\omega)$ is complex, which signals nontrivial capacity and/or inductance.

III. Nontrivial bulk interactions.

1. The massless Thirring model.

$$i(\gamma_t\partial_t - \gamma_x\partial_x)\Psi(t,x,i) =$$

$$\lambda\left[\gamma_t J_t(t,x,i) - \gamma_x J_x(t,x,i)\right]\Psi(t,x,i)\,,$$

where

$$J_\nu(t,x,i) = \overline{\Psi}(t,x,i)\gamma_\nu\Psi(t,x,i)\,.$$

The system is scale invariant and can be quantized by means of our vertex algebra:

$$\Psi_1(t,x,i) = V(t,x,i;(\sigma,\tau))\,,$$

$$\Psi_2(t,x,i) = V(t,x,i;(\tau,\sigma))\,,$$

where

$$\sigma = \sqrt{\frac{\lambda^2}{4}+1}\,, \qquad \tau = -\frac{1}{2}\lambda\,.$$

The conductance tensor reads:

$$G_i^j = \frac{1}{\sqrt{\lambda^2+4}-\lambda}\left(\delta_i^j - S_i^j\right)\,.$$

2. The nonlinear Schrödinger model.

$$\left(\mathrm{i}\partial_t + \partial_x^2\right)\psi(t,x,i) = 2g\,|\psi(t,x,i)|^2\,\psi(t,x,i)\,,$$

In order to incorporate the bulk interaction one has to modify the RT algebra $\mathcal{A}$ as follows:

$$a_{i_1}(k_1)\,a^{*i_2}(k_2)\ -\ a^{*j_2}(k_2)\,R_{i_1 j_2}^{j_1 i_2}(k_1-k_2)\,a_{j_1}(k_1)$$
$$= 2\pi\,\delta(k_1-k_2)\,\delta_{i_1}^{i_2}\,\mathbf{1} + 2\pi\,\delta(k_1+k_2)\,b_{i_1}^{i_2}(k_1)\,,$$

Bulk scattering matrix $R(k)$:

$$R_{12}(k) = \frac{k-\mathrm{i}g}{k+\mathrm{i}g}\,\mathbb{I}_n\otimes\mathbb{I}_n\,,$$

satisfying the Yang-Baxter equation:

$$R_{12}(k_1-k_2)R_{13}(k_1-k_3)R_{23}(k_2-k_3)$$
$$= R_{23}(k_2-k_3)R_{13}(k_1-k_3)R_{12}(k_1-k_2)\,;$$

Boundary (graph vertex) operator $b(k)$:

$$(\Omega_F\,,\,b_i^j(k)\Omega_F) = S_i^j(k)$$

The solution - a sort of operator non-linear Fourier transform (Gelfand, Fokas,...)

$$\psi(t,x,i) = \sum_{m=0}^{\infty} (-g)^m \, \psi_i^{(m)}(t,x)\,,$$

where

$$\psi_i^{(m)}(t,x) =$$

$$\int_{-\infty}^{\infty} \prod_{\substack{k=1\\ l=0}}^{m} \frac{dp_k}{2\pi}\frac{dq_l}{2\pi}\, a^{*i}(p_1)\dots a^{*i}(p_m) a_i(q_m)\dots a_i(q_0)\,\cdot$$

$$\frac{e^{\mathrm{i}\sum\limits_{l=0}^{m}(q_l x - q_l^2 t) - \mathrm{i}\sum\limits_{k=1}^{m}(p_k x - p_k^2 t)}}{\prod\limits_{k=1}^{m}(p_k - q_{k-1} - \mathrm{i}\varepsilon)(p_k - q_k - \mathrm{i}\varepsilon)}\,.$$

is well defined on the finite particle subspace $\mathcal{D}$ in the Fock representation $\mathcal{F}$.

Dependence on g: explicit in the series and implicit in a and a^* through $R(k)$.

On $\mathcal{D}$ the series is actually a finite sum - integral representations of the correlation functions.

V. Conclusions:

(i) Field theory on quantum graphs is an attractive argument both from the physical and mathematical points of view;

(ii) Reflection-transmission algebras represent a natural tool for constructing and studying QFT on graphs both at zero and finite temperature;

(iii) Various physical observables like Casimir energy, Casimir force and conductance are easily derived in this framework;

(iv) Some of the observables have intriguing physical properties - attractive and repulsive forces, enhanced conductance,....

Perspectives:

– Generalize the above framework from star graphs to more general ones;

– Explore more general field theories - fermions, supersymmetric models,...

– Derive the conductance at finite temperature - Kondo effect;

– Construct the analogs of minimal critical models (central charge $0 < c < 1$) on star graphs - Coulomb gas representation on graphs;

– In analogy with string theory, investigate the propagation of graphs on a Riemann space with metric $g^{\mu\nu}$.

$$A = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}t \int_{0}^{\infty} \mathrm{d}x \, g^{\mu\nu}(\varphi_\rho)$$

$$\left[\partial_t \varphi_\mu \partial_t \varphi_\nu - \partial_x \varphi_\mu \partial_x \varphi_\nu\right](t, x, i)$$