Adiabatic Invariance in 2d Euler Flows in Deforming Domains

(INI/HOP 2007-07-03)

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Outline:

- Adiabatic invariants
- Slow Solutions in 2d Euler Flows
- Eulerian picture: Streamfunctions
- Lagrangian picture: Geometric Phase

Adiabatic Invariant Sketch

$$H(p, \varphi; \lambda) = \frac{p^2}{2\lambda^2} - g\lambda \cos \varphi$$

with $\lambda = \lambda(\varepsilon t)$ prescribed.

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$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial \varphi} \qquad \text{and} \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial H}{\partial p}.$$

which are integrable for λ constant

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and change variables $(p, \varphi) \mapsto (I, \theta)$ using a generating function $S(I, \varphi; \lambda)$:

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New Hamiltonian:

$$\tilde{H}(I, \theta; \lambda) = \Omega(I) \qquad \Rightarrow \qquad \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\partial \tilde{H}}{\partial \theta}$$

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2d Euler Flow

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} p = 0$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

plus BC: $\boldsymbol{v} \cdot \boldsymbol{n} = \varepsilon \beta(\varepsilon t)$ on $\partial D_{\Lambda(\varepsilon t)}$ with Λ and β prescribed.

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Use vorticity $\omega = \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{v} := \partial_x v - \partial_y u$ formulation:

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Vorticity is materially conserved (thanks to Hamiltonian structure of 2dE):

$$\mathcal{C}_f = \int_{D_{\Lambda}} f(\omega) \, \mathrm{d}x \, \mathrm{d}y$$

is constant for any integrable f.

Slow Solutions

Consider dynamics in slow time $\tau = \varepsilon t$

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$$\omega = \omega^{(0)} + \varepsilon\omega^{(1)} + \varepsilon^2\omega^{(2)} + \cdots \qquad \text{and} \qquad \psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots$$

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$$\partial(\psi^{(0)}, \omega^{(0)}) + \varepsilon\partial_\tau\omega^{(0)} + \varepsilon\partial(\psi^{(1)}, \omega^{(0)}) + \varepsilon\partial(\psi^{(0)}, \omega^{(1)}) + \mathcal{O}(\varepsilon^2) = 0$$
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Assumptions:

H0. $D_{\Lambda(\varepsilon t)}$ is smooth and simply connected

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H2. Timescale separation: there exists $c_\psi > 0$ s.t. for all relevant c,

$$\oint_{\psi^{(0)}=c} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi^{(0)}|} \leq \frac{1}{c_\psi}$$

Kinematic Streamfunction ϕ

$$\partial_\tau \omega^{(0)} + \partial(\phi, \omega^{(0)}) = 0$$

Write $\psi^{(0)} = G(\omega^{(0)}; \tau)$ as $\psi_\Lambda = G_{\Lambda(\tau)}(\omega_\Lambda)$ and take ∂_τ,

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where $F_\Lambda \circ G_\Lambda = 1$ and $(\partial_\tau G_\Lambda) \circ \omega_\Lambda = \partial_\tau \chi$

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We need to solve (*) with constraint and BC

$$\partial(\chi, \omega_\Lambda) = 0$$

$$\phi = b \quad \text{and} \quad \chi = 0 \qquad \text{on } \partial D_\Lambda$$

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Using the projection

$$\mathsf{P}_\Lambda u := u - \Big\{ \oint_{\psi_\Lambda = c} u \, \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|} \Big\} \Big/ \Big\{ \oint_{\psi_\Lambda = c} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|} \Big\}$$

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which is solvable when $\Delta - F'_\Lambda \mathsf{P}_\Lambda$ is coercive (implied by H1)

Persistence of Steady Flows

Regarding steady flows in 2d Euler as (relative) equilibria of a dynamical system, it can be shown [W & Vanneste 05] that

> Given $\psi_0 \in C^{k,\alpha}(D_0)$ and $D_{\Lambda(\tau)} \in C^{k,\alpha}$ with $k \geq 3$ and $\alpha > 0$, ψ_Λ remains steady as long as H1 and H2 hold

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Other approaches:

- Arnold & Khesin: does not conserve vorticity
- Shepherd, Moffatt: always find a steady flow, not necessarily nearby

Example: Slightly Deformed Disc

$$\psi_0 = r^{1/2}$$

$$r = 1 + \delta \sum_m \Lambda_m \mathrm{e}^{\mathrm{i}m\sigma} - \frac{\delta^2}{2} \sum_m |\Lambda_m|^2 + \mathcal{O}(\delta^3)$$

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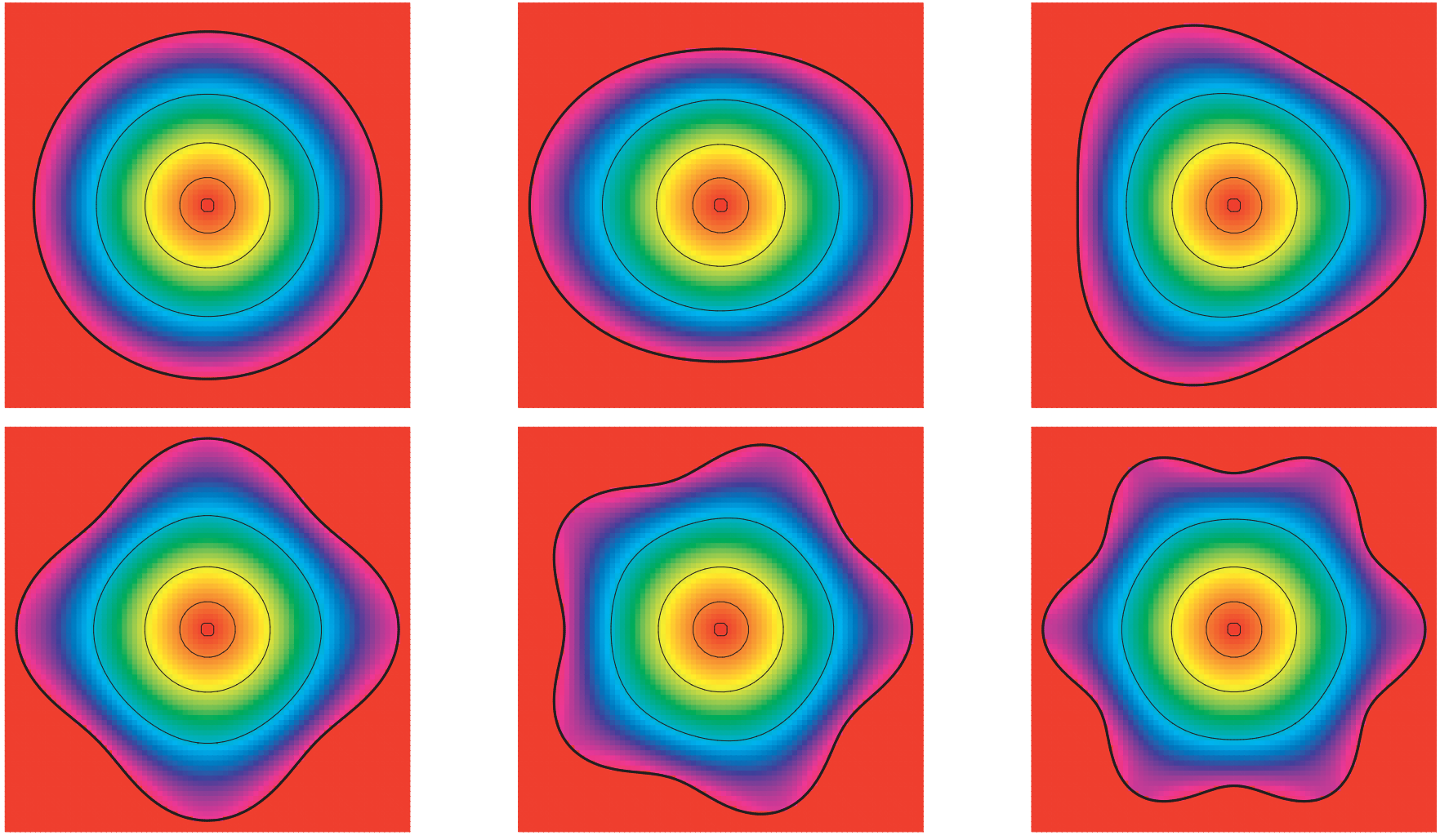

Dynamic Streamfunctions $\psi^{(i)}$

$$(\Delta - F'_\Lambda \mathsf{P}_\Lambda)\varphi = 0$$
$$\varphi = \partial(\psi_\Lambda, b) \qquad \text{on } \partial D_\Lambda$$

Since both ω and ω_Λ are rearranged, $\mathsf{P}_\Lambda \omega^{(1)} = \omega^{(1)}$ and

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In principle, this can be continued to higher orders: eg,

$$(\Delta - F'_\Lambda)\psi^{(2)} = \Phi_\Lambda(\psi^{(1)}) + \xi \circ \omega_\Lambda$$

plus an integral constraint on $\psi^{(2)}$ to fix ξ,

$$\oint_{\psi_\Lambda} F'_\Lambda \psi^{(2)} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|} = -\oint_{\psi_\Lambda} \left\{ \Phi_\Lambda + \omega^{(1)} \frac{\boldsymbol{\nabla}\omega_\Lambda \cdot \boldsymbol{\nabla}\omega^{(1)}}{|\boldsymbol{\nabla}\omega_\Lambda|^2} \right\} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|}$$

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Thus $[\cdots\cdots]\ \psi^{(i)} = 0$ when $b = 0$ for $i = 1, 2, \cdots$

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In principle, this can be continued to higher orders: eg,

$$(\Delta - F'_\Lambda)\psi^{(2)} = \Phi_\Lambda(\psi^{(1)}) + \xi \circ \omega_\Lambda$$

plus an integral constraint on $\psi^{(2)}$ to fix ξ,

$$\oint_{\psi_\Lambda} F'_\Lambda \psi^{(2)} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|} = -\oint_{\psi_\Lambda} \left\{ \Phi_\Lambda + \omega^{(1)} \frac{\boldsymbol{\nabla}\omega_\Lambda \cdot \boldsymbol{\nabla}\omega^{(1)}}{|\boldsymbol{\nabla}\omega_\Lambda|^2} \right\} \frac{\mathrm{d}l}{|\boldsymbol{\nabla}\psi_\Lambda|}$$

Thus $[\cdots\cdots]\ \psi^{(i)} = 0$ when $b = 0$ for $i = 1, 2, \cdots$

$\Rightarrow$ In 2d Euler, ψ_Λ is a "slow manifold" (we don't know if it's stable)

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Particle position is governed by the Hamiltonian system

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$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\varepsilon \frac{\partial}{\partial \theta} \mathcal{F}(\cdots) \cdot \frac{\mathrm{d}\Lambda}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial \tilde{\psi}_\Lambda}{\partial I} + \varepsilon \frac{\partial}{\partial I} \mathcal{F}(\cdots) \cdot \frac{\mathrm{d}\Lambda}{\mathrm{d}t}$$

Geometric Phase

If Λ traces a closed loop $\Gamma = \partial\mathcal{S}$ in parameter space, the second term in $\Delta\theta$ can be given a purely geometric interpretation:

$$(*) \qquad \frac{\partial}{\partial I}\int \mathcal{F}(\cdots)\cdot\frac{\mathrm{d}\Lambda}{\mathrm{d}t}\,\mathrm{d}t = \frac{\partial}{\partial I}\int_{\mathcal{S}}\left\{\mathbf{d}\Phi^{\star} + \tfrac{1}{2}[\Phi^{\star}\wedge\Phi^{\star}]\right\} + \mathcal{O}(\varepsilon)$$

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The integrand $\Phi^\star + \frac{1}{2}[\Phi^\star\wedge\Phi^\star]$ is a curvature form, as with Hannay & Berry

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