Tree shapes
- induced by neutral models for speciation -

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The constant rate birth- and death processes
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- Birth rate $\lambda$.
- Death rate $\mu$. 
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- We observe $n$ species today.
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Reconstructed trees

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time

1  2  3  4  5
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Reconstructed trees

- Point Process representation.
Reconstructed trees

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- Speciation times $s_i$ i.i.d.
Reconstructed trees

- Point Process representation.
- Speciation times $s_i$ i.i.d.

\[
f(s_i | \lambda, \mu, t_{or} = t, n) = (\lambda - \mu)^2 \frac{e^{-(\lambda-\mu)s_i}}{(\lambda - \mu e^{-(\lambda-\mu)s_i})^2} \frac{\lambda - \mu e^{-(\lambda-\mu)t}}{1 - e^{-(\lambda-\mu)t}}
\]
Successive speciation events

Let $\mathcal{A}_n^k$ be the time of the $k$-th speciation event in a phylogeny with $n$ extant species, time of origin uniform prior.
Successive speciation events

Let $A_n^k$ be the time of the $k$-th speciation event in a phylogeny with $n$ extant species, time of origin uniform prior.

\[
f_{A_n^k}(s) = (k + 1) \binom{n}{k+1} \lambda^{n-k} (\lambda - \mu)^{k+2} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad e^{-(\lambda-\mu)(k+1)s} \frac{1 - e^{-(\lambda-\mu)s} n-k-1}{(\lambda - \mu e^{-(\lambda-\mu)s} n+1}
\]
Successive speciation events

Let $A^k_n$ be the time of the $k$-th speciation event in a phylogeny with $n$ extant species, time of origin uniform prior.

$$f_{A^k_n}(s) = (k + 1) \binom{n}{k + 1} \lambda^{n-k} (\lambda - \mu)^{k+2} e^{-(\lambda-\mu)(k+1)s} \frac{(1 - e^{-(\lambda-\mu)s})^{n-k-1}}{(\lambda - \mu e^{-(\lambda-\mu)s})^{n+1}}$$

in particular:

$$f^{Yule}_{A^k_n}(s) = (k + 1) \binom{n}{k + 1} \lambda \frac{(e^{\lambda s} - 1)^{n-k-1}}{e^{\lambda s n}}$$

$$f^{CBP}_{A^k_n}(s) = (k + 1) \binom{n}{k + 1} \lambda^{n-k} \frac{s^{n-k-1}}{(1 + \lambda s)^{n+1}}$$
Expectation of successive speciation events

\[
\mathbb{E}[\mathcal{A}_n^k] = (k + 1) \binom{n}{k + 1} (-1)^k \sum_{i=0}^{n-k-1} \binom{n - k - 1}{i} \frac{(\lambda - \mu)^{k+i}}{k+i+1} \frac{1}{\mu^{k+i+1}} \ldots 
\]

\[
\left[ \log \left( \frac{\lambda}{\lambda - \mu} \right) - \sum_{j=1}^{k+i} \binom{k+i}{j} \frac{(-1)^j}{j} \left( 1 - \left( \frac{\lambda}{\lambda - \mu} \right)^j \right) \right]
\]
Expectation of successive speciation events

\[ E[A^k_n] = (k + 1) \binom{n}{k + 1} (-1)^k \sum_{i=0}^{n-k-1} \binom{n - k - 1}{i} \frac{(\lambda - \mu)^{k+i}}{k + i + 1} \frac{1}{\mu^{k+i+1}} \ldots \]

\[ \left[ \log \left( \frac{\lambda}{\lambda - \mu} \right) - \sum_{j=1}^{k+i} \binom{k + i}{j} \frac{(-1)^j}{j} \left( 1 - \left( \frac{\lambda}{\lambda - \mu} \right)^j \right) \right] \]

in particular:

\[ E_{Yule}[A^k_n] = \sum_{i=k+1}^{n} \frac{1}{\lambda i} \]

\[ E_{CBP}[A^k_n] = \frac{n - k}{\lambda k} \]
Given the tree shape, we obtained the displayed speciation times under the CBP (left) and Yule model (right).
Expected LTT plots ("horizontal")

$n = 10$.
Most recent common ancestor is at 0 and today is 1.
Expected LTT plots ("vertical")

\( n = 10, \quad t = 10. \)
\( \lambda = 5, 2, 1, 0.5, 0.2, 0.1, 0.01, \) from bottom to top.
green: \( \mu = 0, \) yellow: \( \mu = \lambda/4, \) blue: \( \mu = \lambda/2, \) red: \( \mu = 3/4\lambda, \) black: \( \mu = \lambda. \)
$n = 10$.
Expectation only depends on $\rho = \mu/\lambda$.
$\rho = 0, 1/4, 1/2, 3/4, 1$ (from top to bottom).
Dankeschön

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