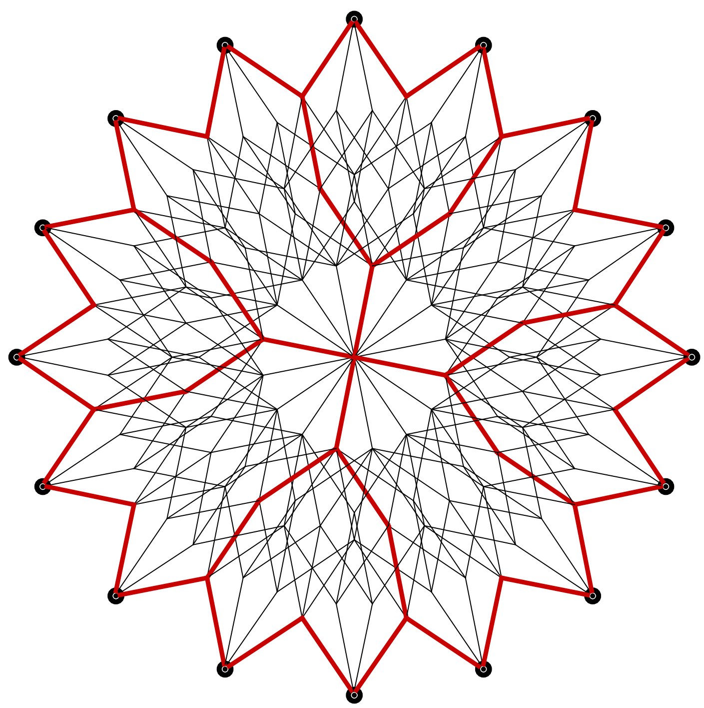
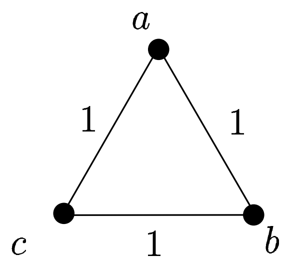
Optimal and Hereditarily Optimal Realizations of Metric Spaces

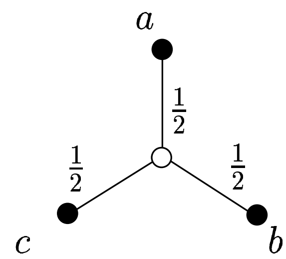
Alice Lesser, Uppsala Universitet, Sweden

Joint work with Vincent Moulton, UEA, UK
and Jack Koolen, POSTECH, South Korea

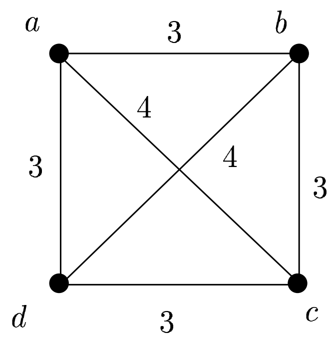


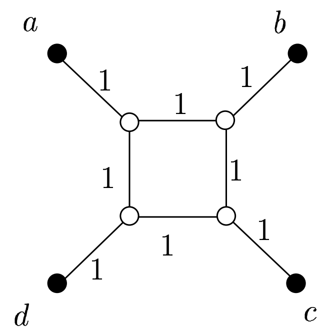
$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$



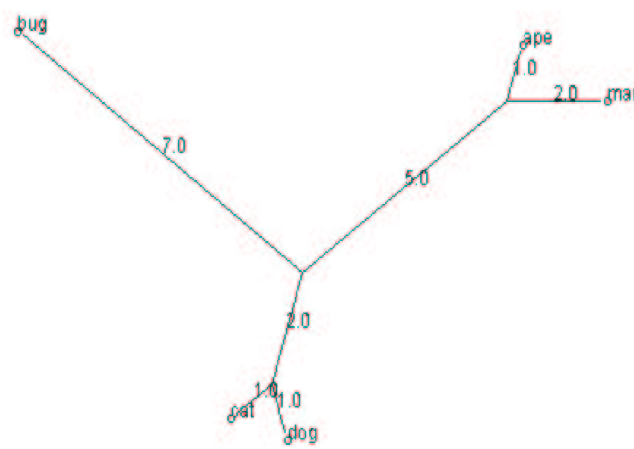


$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 3 & 4 \\ 4 & 3 & 0 & 3 \\ 3 & 4 & 0 & 4 \end{bmatrix} \end{array}$$

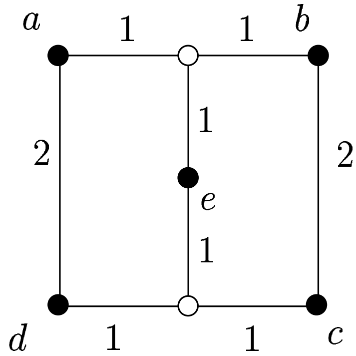


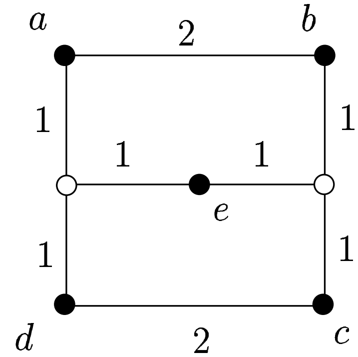


$$\begin{array}{c} man \\ ape \\ dog \\ cat \\ bug \end{array} \begin{bmatrix} 0 & 3 & 10 & 10 & 14 \\ 3 & 0 & 9 & 9 & 13 \\ 10 & 9 & 0 & 2 & 10 \\ 10 & 9 & 2 & 0 & 10 \\ 14 & 13 & 10 & 10 & 0 \end{bmatrix}$$



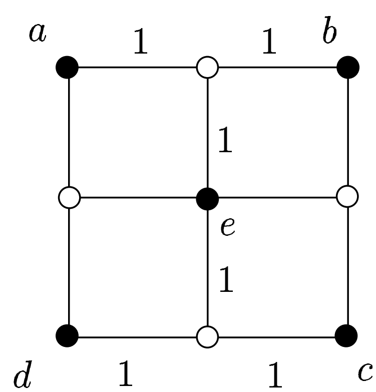
	a	b	c	d	e
a	0	2	4	2	2
b	2	0	2	4	2
c	4	2	0	2	2
d	2	4	2	0	2
e	2	2	2	2	0



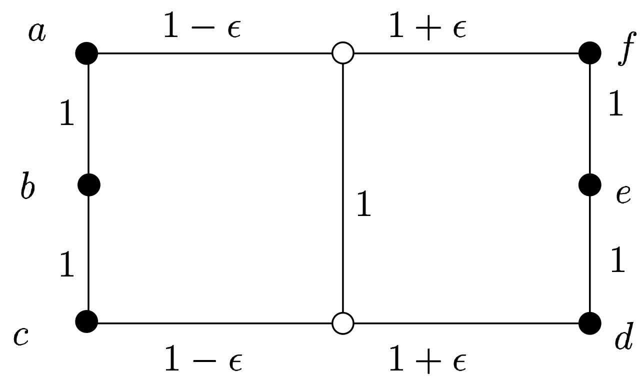


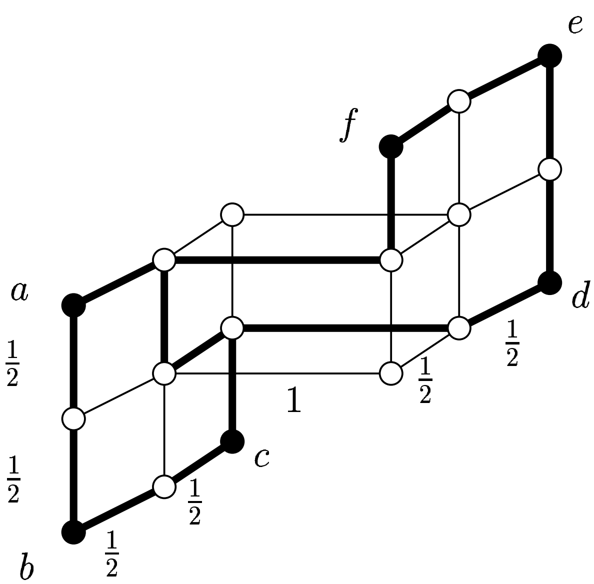
$|X| \le 2$: any optimal realization of d is h-optimal.
$|X| \ge 3$: a realization $G = (V, E, w)$ of (X, d) is h-optimal if for any $Y \subsetneq X$ there is some subgraph $G' = (V', E', w\,|_{E'})$ of G such that G' is an h-optimal realization of $(Y, d|_Y)$ and if $\sum_{e \in E} w(e)$ is minimal among all such graphs.

	a	b	c	d	e
a	0	2	4	2	2
b	2	0	2	4	2
c	4	2	0	2	2
d	2	4	2	0	2
e	2	2	2	2	0

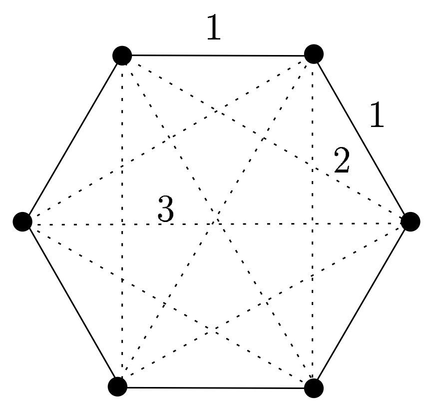


	a	b	c	d	e	f
a	0	1	2	3	3	2
b	1	0	1	3	4	3
c	2	1	0	2	3	3
d	3	3	2	0	1	2
e	3	4	3	1	0	1
f	2	3	3	2	1	0





The *underlying graph* $UG(d) = (V, E, w)$ associated to (X, d) is the graph with
$V = X$,
$E = \{\{x, y\} \in \binom{X}{2}$ with $d(x, z) + d(z, y) > d(x, y)\}$,
$w(\{x, y\}) = d(x, y)$.



Theorem 1. *Suppose that (X, d) is a finite metric space. Then $UG(d)$ is an optimal realization of d if and only if the following two conditions hold:*

(i) for every pair of edges $\{x, y\}, \{y, z\}$ in $UG(d)$ with x, z distinct, $d(x, z) = d(x, y) + d(y, z)$.

(ii) for every disjoint pair of edges $\{x, y\}, \{u, v\}$ in $UG(d)$, either $d(x, y)+d(u, v) < \max\{d(x, u)+d(y, v), d(x, v)+d(y, u)\}$ or $d(x, y) + d(u, v) = d(x, u) + d(y, v) = d(x, v) + d(y, u)$.

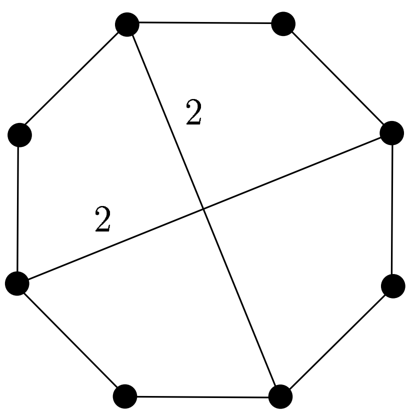
A *Hadamard graph of valency* $2m$ is a connected, $2m$-regular, bipartite, antipodal graph with diameter 4.

For a Hadamard graph H_k with valency 2^k, $(k \geq 1)$ and vertex set X_k, we define a metric d_k on X_k:

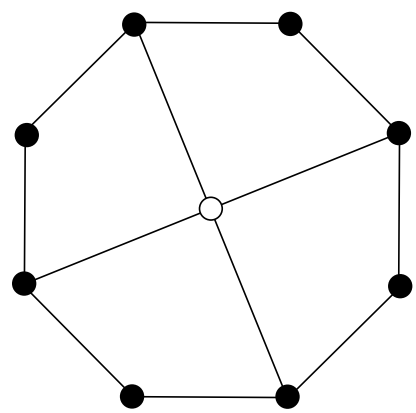
Fix a vertex $x \in X_k$, and let

$$Y_k = \{y \,:\, d_{H_k}(x, y) \in \{0, 2, 4\}\},$$

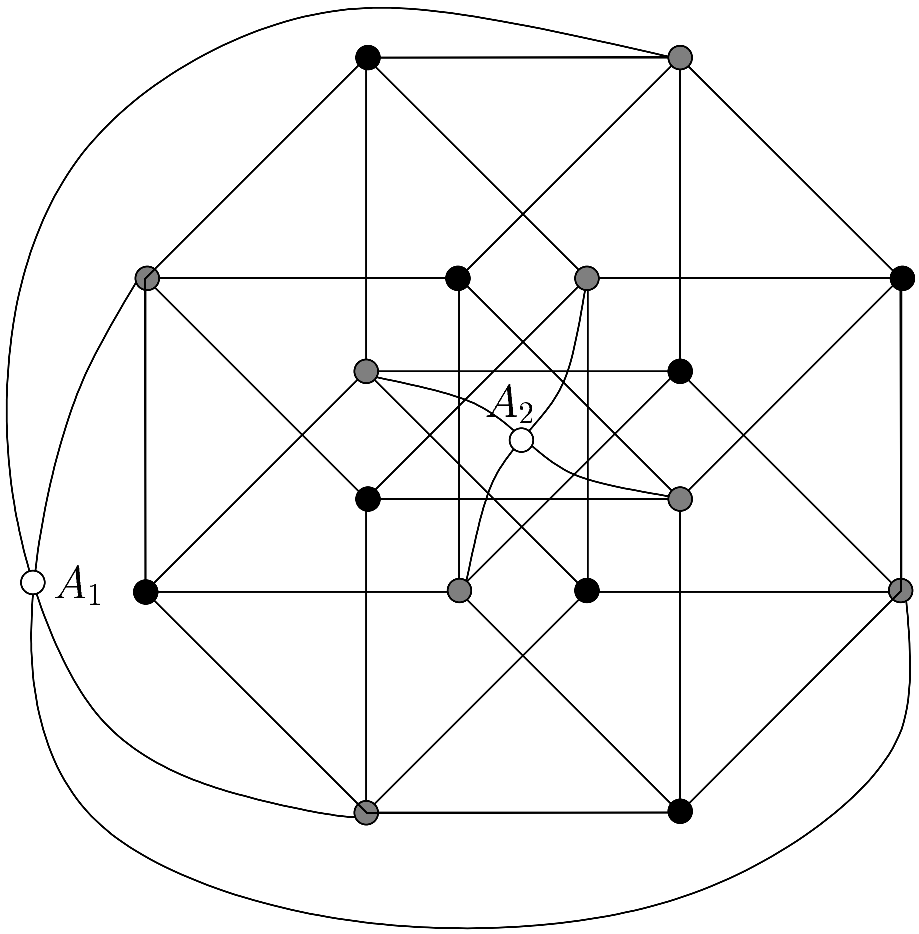
$$d_k(x, y) = \begin{cases} 2 & \text{if } x \in Y_k \text{ and } y = \overline{x}, \\ d_{H_k}(u, v) & \text{else.} \end{cases}$$



$UG(d_8)$



h-optimal realization


A_2
A_1