On correlation functions of quantum integrable models

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The Heisenberg spin chain

• **Model for magnetism in solids (Heisenberg, 1928)**
  ★ Crystals with effective one-dimensional magnetic properties
  ★ Can be tested via inelastic neutron scattering experiments

• **Archetype of quantum integrable models**
  ★ Spectrum resolution via Bethe ansatz (1931) and its developments
  ★ Links to two-dimensional statistical mechanics (vertex models generalizing Ising)

• **Very rich (non-commutative) algebraic structures**
  ★ Yang-Baxter algebras, R-matrices, Quantum groups
  ★ They appear in different situations eventually far from magnetism (Gauge and String theories and AdS/CFT correspondence)
  ★ Link to combinatorics in special point (ice model)
Crystals with one-dimensional magnetic behavior

There exists quite many (three-dimensional) crystals with magnetic properties well described by one-dimensional Heisenberg models (ferromagnetic or anti-ferromagnetic):

- One-dimensional structure due to magnetic interactions along chains of magnetic ions, while inter-chain magnetic coupling is damped by the presence of other large non-magnetic ions or complexes (TMMC, CuCl2.2NC5H5, CoCl2.2NC5H5, ...)

- More exotic one-dimensional structure due to other effects, like the Jahn-Teller effect in KCuF3.
Correlation functions

Type (a)  Type (d)

Cu
F

K
Neutron scattering experiments and correlation functions

\[ S^{\alpha\beta}(m, t) = \frac{\text{tr} (\sigma^\alpha_1 e^{iHt} \sigma^{\beta}_{m+1} e^{-iHt} e^{-\frac{H}{kT}})}{\text{tr} (e^{-\frac{H}{kT}})}. \]  

(1)

At zero temperature, this expression reduces to an average value of the product of Heisenberg spin operators taken in the ground state \(|\psi_g\rangle\), the normalized (non degenerated in the disordered regime) state with lowest energy level of the Heisenberg chain,

\[ S^{\alpha\beta}(m, t) = \langle \psi_g \mid \sigma^\alpha_1 e^{iHt} \sigma^{\beta}_{m+1} e^{-iHt} \mid \psi_g \rangle. \]  

(2)

For example the longitudinal structure factor is defined as the Fourier transform (in space and time) of the dynamical correlation functions,

\[ S^{zz}(q, \omega) = \frac{1}{N} \sum_{j, j'=1}^{N} e^{iq(j-j')} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle S^z_j(t) S^z_{j'}(0) \rangle. \]
is related, at first order in the neutron-crystal interaction, to the differential magnetic cross sections for the inelastic scattering of unpolarized neutrons off a crystal (like \( KCuF_3 \)), with energy transfer \( \omega \) and momentum transfer \( q \) through the following formula:

\[
\frac{d\sigma}{d\Omega d\omega} \sim (\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}) S^{\alpha\beta}(q, \omega)
\]

(3)
$S(Q, \omega)$ is the Fourier transform of the dynamical spin-spin correlation function. The Bethe ansatz curve (on the left) is computed here for a chain of 500 sites while the experimental curve obtained by A. Tennant and his team in Berlin by neutron scattering is presented on the right. Colors indicate the height of the function $S(Q, \omega)$. 
The spin-1/2 XXZ Heisenberg chain

The $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field is a quantum interacting model defined on a one-dimensional lattice with $M$ sites, with Hamiltonian, $H = H^{(0)} - \hbar S_z$,

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1) \right\},$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \quad [H^{(0)}, S_z] = 0.$$

$\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site $m$ of the chain.

Quantum space of states : $\mathcal{H} = \bigotimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$ act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.
Correlation functions of Heisenberg chain

- **Free fermion point** $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa, ...

- **From 1984**: Izergin, Korepin, ... (first attempts using Bethe ansatz for general $\Delta$)

- **General $\Delta$**: multiple integral representations
  - 1996 Jimbo and Miwa → from qKZ equation
  - 1999 Kitanine, Maillet, Terras → from Algebraic Bethe Ansatz

- **Several developments since 2000**: Kitanine, Maillet, Slavnov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov, Takeyama; Gohmann, Klumper, Seel; Caux, Hagemans, Maillet; ...
Correlation functions

\[ \langle \mathcal{O} \rangle = \frac{\text{tr}_\mathcal{H}(\mathcal{O} e^{-H/kT})}{\text{tr}_\mathcal{H}(e^{-H/kT})} \]

\[ = \langle \omega | \mathcal{O} | \omega \rangle \quad \text{at} \quad T = 0 \]

where \( |\omega\rangle \) is the state with lowest eigenvalue. Why is it so difficult? (Bethe ansatz already 75 years old...!)

Consider the correlation function of the product of two local operators, \( \mathcal{O} = \theta_1 \theta_2 \) at zero temperature:

\[ g_{12} = \langle \omega | \theta_1 \theta_2 | \omega \rangle \]

Two main strategies to evaluate such a function:

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(i) compute the action of local operators on the ground state $\theta_1 \theta_2 |\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle \omega | \tilde{\omega} \rangle$$

(ii) insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion):

$$g_{12} = \sum_i \langle \omega | \theta_1 | \omega_i \rangle \cdot \langle \omega_i | \theta_2 | \omega \rangle$$

Main problems to be solved to achieve this:

- Compute exact eigenstates and energy levels of the Hamiltonian (Bethe ansatz)
- Obtain the action of local operators on the eigenstates: main problem since eigenstates are highly non-local!
- Compute the resulting scalar products with the eigenstates
The methods...

- **q-KZ and q-vertex operators**:  
  - Valid (with some hypothesis) for infinite (and semi-infinite) chains, zero magnetic field and zero temperature  
  - Elementary blocks of correlation functions (static) and form factors (massive case)  
  - Multiple integrals and recently algebraic solutions of q-KZ

- **Bethe ansatz**:  
  - Valid for finite and infinite chains, with magnetic field and temperature, and with impurities or with integrable boundaries (to be published)  
  - Elementary blocks of correlation functions, spin-spin, dynamical case, form factors  
  - Determinant representation of form factors (finite chain), multiple integrals for correlation functions (infinite chain), master formula for spin-spin correlation functions.  
  - Some results for a continuum model (NLS)
Algebraic Bethe ansatz and correlation functions

- **Algebraic Bethe ansatz** (Faddeev, Sklyanin, Taktadjan)

  Construction of the direct map: \( \sigma^\alpha_m \mapsto T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \)

  \( T(\lambda) \equiv T_{a,1...N}(\lambda) = L_{aN}(\lambda - \xi_N) \ldots L_{a1}(\lambda - \xi_1) \)

  \( L_{an}(\lambda) \) being \( 2 \times 2 \) matrices with entries function of \( \sigma^x_{n,y,z} \) operators in site \( n \).

  **Yang-Baxter algebra**: \( R_{12}(\lambda_1, \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1, \lambda_2) \)

  **Commuting conserved charges**: \( t(\lambda) = A(\lambda) + D(\lambda), \quad [t(\lambda), t(\mu)] = 0 \)

  **Hamiltonian**: \( H = 2 \sinh \eta \frac{\partial}{\partial \lambda} \log t(\lambda) \bigg|_{\lambda = \frac{\eta}{2}} + c \) for all \( \xi_j = 0 \).

  **Eigenstates of** \( t(\mu) \): \( |\psi\rangle = \prod_k B(\lambda_k)|0\rangle \) with \( \{\lambda_k\} \) solution of the Bethe equations.
Action of local operators on Bethe states (Kitanine, Maillet, Terras)

Solution of the quantum inverse scattering problem: $\sigma_{\alpha}^\alpha \longleftarrow T(\lambda)$

\begin{align*}
\sigma_j^- &= \prod_{k=1}^{j-1} t(\xi_k) \cdot B(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\
\sigma_j^+ &= \prod_{k=1}^{j-1} t(\xi_k) \cdot C(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\
\sigma_j^z &= \prod_{k=1}^{j-1} t(\xi_k) \cdot (A - D)(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k),
\end{align*}

(4)

+ Yang-Baxter algebra for A, B, C, D to get the action on arbitrary states, for example

$$
\langle 0 \vert \prod_{k=1}^{N} C(\lambda_k) A(\lambda_{N+1}) = \sum_{a'=1}^{N+1} \Lambda_{a'} \langle 0 \vert \prod_{k=1}^{N+1} C(\lambda_k)
$$
• Scalar products (Slavnov; Kitanine, Maillet, Terras)

\[
\langle 0 | \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{N} B(\lambda_k) | 0 \rangle = \frac{\det U(\{\mu_j\}, \{\lambda_k\})}{\det V(\{\mu_j\}, \{\lambda_k\})}
\]

for \(\{\lambda_k\}\) a solution of Bethe equations and \(\{\mu_j\}\) an arbitrary set of parameters, :

\[
U_{ab} = \partial_{\lambda_a} \tau(\mu_b, \{\lambda_k\}), \quad V_{ab} = \frac{1}{\sinh(\mu_b - \lambda_a)}, \quad 1 \leq a, b \leq N,
\]

where \(\tau(\mu_b, \{\lambda_k\})\) is the eigenvalue of the transfer matrix \(t(\mu_b)\)
Quantum groups and the space of states

The operators $A$, $B$, $C$, $D$ are highly non local in terms of local spin operators. There exists however a change of basis of the space of states provided by the notion of Drinfel’d twist associated to the $R$-matrix of the $XXZ$ chain and to the quantum affine algebra $\mathcal{U}_q(\widehat{sl}_2)$.

For inhomogeneity parameters $\xi_j$ in generic positions and to any element $\sigma$ of the symmetric group $S_n$ there is a unique $R^\sigma_{1\ldots n}$ matrix, constructed as an ordered product (depending on $\sigma$) of the elementary $R$-matrices $R_{ij}(\xi_i, \xi_j)$.

$$R^\sigma_{1\ldots n} T_{1\ldots n}(\lambda; \xi_1, \ldots, \xi_n) = T_{\sigma(1)\ldots \sigma(n)}(\lambda; \xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}) R^\sigma_{1\ldots n}. $$

Factorizing $F_{1\ldots n}(\xi_1, \ldots, \xi_n)$ matrix :

$$F_{\sigma(1)\ldots \sigma(n)}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}) R^\sigma_{1\ldots n}(\xi_1, \ldots, \xi_n) = F_{1\ldots n}(\xi_1, \ldots, \xi_n). $$
Simplified notations:

\[ F_{1\ldots n}(\xi_1, \ldots, \xi_n) = F_{1\ldots n}, \]
\[ F_{\sigma(1)\ldots\sigma(n)}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}) = F_{\sigma(1)\ldots\sigma(n)}. \]

In the \( F \)-basis, the monodromy matrix \( \tilde{T} \)

\[ \tilde{T}_{1\ldots M}(\lambda; \xi_1, \ldots, \xi_M) = F_{1\ldots M} T_{1\ldots M}(\lambda; \xi_1, \ldots, \xi_M) F_{1\ldots M}^{-1}, \]

is totally symmetric under any simultaneous permutations of the lattice sites \( i \) and of the corresponding inhomogeneity parameters \( \xi_i \).

The quantum monodromy operator is a \( 2 \times 2 \) matrix with entries \( A, B, C, D \) which are highly non-local. As an example, the \( B \) operator is given as

\[ B_{1\ldots M}(\lambda) = \sum_{i=1}^{N} \sigma_i^- \Omega_i + \sum_{i \neq j \neq k} \sigma_i^- (\sigma_j^- \sigma_k^+) \Omega_{ijk} + \text{higher terms}, \]
The operators \( D, B \) and \( C \) in the \( F \)-basis are given by the (quasi-local) formulas:

\[
\tilde{D}_{1\ldots M}(\lambda; \xi_1, \ldots, \xi_M) = \bigotimes_{i=1}^{M} \left( \begin{array}{cc} b(\lambda, \xi_i) & 0 \\ 0 & 1 \end{array} \right)_{[i]}
\]

\[
\tilde{B}_{1\ldots M}(\lambda) = \sum_{i=1}^{M} \sigma_i^{-} c(\lambda, \xi_i) \bigotimes_{j \neq i} \left( \begin{array}{cc} b(\lambda, \xi_j) & 0 \\ 0 & b^{-1}(\xi_j, \xi_i) \end{array} \right)_{[j]}
\]

\[
\tilde{C}_{1\ldots M}(\lambda) = \sum_{i=1}^{M} \sigma_i^{+} c(\lambda, \xi_i) \bigotimes_{j \neq i} \left( \begin{array}{cc} b(\lambda, \xi_j) & b^{-1}(\xi_i, \xi_j) \\ 0 & 1 \end{array} \right)_{[j]}
\]

and the operator \( \tilde{A} \) can be obtained from quantum determinant relations.

It really means that the factorizing \( F \)-matrices we have constructed solve the combinatorial problem induced by the non-trivial action of the permutation group \( S_M \) given by the \( R \)-matrix. In the \( F \)-basis the action of the permutation group on the operators \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) is trivial.
Matrix elements of local operators

For example:

\[
\langle 0| \prod_{j=1}^{N} C(\mu_j) \sigma_n^z \prod_{k=1}^{N} B(\lambda_k) |0 \rangle =
\]

\[
= \langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{n-1} t(\xi_k) \cdot (A - D)(\xi_j) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k) \prod_{k=1}^{N} B(\lambda_k) |0 \rangle
\]

Here the sets \{\lambda_k\} and \{\mu_j\} are both solutions of Bethe equations

\[
\langle 0| \prod_{j=1}^{N} C(\mu_j) \sigma_n^z \prod_{k=1}^{N} B(\lambda_k) |0 \rangle = \Phi_n \langle 0| \prod_{j=1}^{N} C(\mu_j) (A - D)(\xi_j) \prod_{k=1}^{N} B(\lambda_k) |0 \rangle
\]

Hence it leads to determinant representations of these matrix elements (using the scalar product formula)
Spontaneous magnetisation

The ground state of the XXZ model in the region $\Delta = \frac{1}{2}(q + q^{-1}) > 1$, $q > 1$, is degenerated in the thermodynamic limit ($M \to \infty$): the ground state $|\Psi_1\rangle$ and the quasi-ground state $|\Psi_2\rangle$ (on the finite lattice, these states possess different energies).

The spontaneous magnetization:

$$s_0 = \left| \frac{\langle \Psi_1 | \sigma^z_m | \Psi_2 \rangle}{\langle \Psi_1 | \Psi_1 \rangle^{1/2} \langle \Psi_2 | \Psi_2 \rangle^{1/2}} \right|.$$

Baxter formula:

$$s_0 = \left( \prod_{n=1}^{\infty} \frac{1 - q^{-2n}}{1 + q^{-2n}} \right)^2, \quad (M = \infty).$$

This formula in the thermodynamic limit was also reproduced by means of the q-vertex operator approach. Here there is a direct derivation + control of finite size corrections. Using the determinant formula we get it as the infinite product of the corresponding matrix eigenvalues.
Analytical + Numerical methods for dynamical correlation functions in a field

(Caux, Hagemans, Maillet)

We consider finite chain of length $N$, and a ground state depending on the magnetic field, with a fixed number of reversed spins $M$, number of sites $N$ even, and $2M \leq N$. We use form factor expansion, namely we sum over a complete set of intermediate eigenstates $\langle \omega |$ to obtain:

$$g_{12} = \sum_i \langle \omega | \theta_1 | \omega_i \rangle \cdot \langle \omega_i | \theta_2 | \omega \rangle$$

Then each term is given by an explicit determinant of size $M$, depending on two sets of parameters solutions of Bethe equations and characterizing the states $\langle \omega |$ and $| \omega_i \rangle$ respectively.

Numerics are then used to compute the determinants and the (finite) sum to get the dynamical correlation functions (control of the results via sum rules): successful comparison to neutron scattering experiments performed on various compounds.
Correlation functions: elementary blocks

\[ F_m(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle \psi_g | \prod_{j=1}^{m} E_j^{\epsilon'_j, \epsilon_j} | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle} \quad E_l^{\epsilon', \epsilon} = \delta_{l, \epsilon'} \delta_{k, \epsilon} \quad (5) \]

Solution of the quantum inverse scattering problem + Yang-Baxter algebra of operators

\[ T(\lambda) \rightarrow \text{Multiple integral formula for the correlation functions for infinite lattice} \]

\[ F_m(\{\epsilon_j, \epsilon'_j\}) = (\prod_{k=1}^{m} \int_{C^h_k} d\lambda_k) \Omega_m(\{\lambda_k\}, \{\epsilon_j, \epsilon'_j\}) S_h(\{\lambda_k\}) \quad (6) \]

where \( \Omega_m(\{\lambda_k\}, \{\epsilon_j, \epsilon'_j\}) \) is purely algebraic and \( S_h(\{\lambda_k\}) \), \( C^h_k \) are depending on the regime and the magnetic field \( h \).

\[ \rightarrow \text{Proof of the results and conjectures of Jimbo, Miwa et al. and extension to the non zero magnetic field } h \text{ (a case where the quantum affine symmetry is broken); more recently, extension to time dependent and non zero temperature cases and to open chains} \]
What about this result?

→ A priori, the problem is solved:
  • expression of all elementary blocks \( \langle \psi_g | E_{1,1}^{\epsilon_1} \ldots E_{m,m}^{\epsilon_m} | \psi_g \rangle \)
  • any correlation function = \( \sum \) (elementary blocks)

→ From a practical point of view, there are **two main problems:**
  
  (1) physical correlation function = huge sum of elementary blocks at large distances
  
  Example: **two-point function**

\[
\langle \psi_g | \sigma_1^{\hat{z}} \sigma_m^{\hat{z}} | \psi_g \rangle \equiv \langle \psi_g | (E_{1,1}^{11} - E_{1,1}^{22}) \prod_{j=2}^{m-1} (E_{j,1}^{11} + E_{j,1}^{22}) (E_{m,1}^{11} - E_{m,1}^{22}) | \psi_g \rangle
\]

\[
= \sum_{2^m \text{ terms}} \text{(elementary blocks)} \quad \sim \quad m \to \infty ?
\]

\( \sim \) re-summation: needed to challenge the contact with conformal field theory!
(2) each block has a complicated expression
Example: emptiness formation probability for $h = 0$ in the massless regime ($-1 < \Delta = \cosh \zeta < 1$)

$$
\tau(m) \equiv \langle \psi_g | \prod_{k=1}^{m} \frac{1 - \sigma_k^z}{2} | \psi_g \rangle
$$

$$
= (-1)^m \left( - \frac{\pi}{\zeta} \right)^{m(m-1)/2} \int_{-\infty}^{\infty} \frac{d^m \lambda}{2\pi} \prod_{a>b}^{m} \frac{\sinh \frac{\pi}{\zeta}(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b - i\zeta)}
$$

$$
\times \prod_{j=1}^{m} \frac{\sinh^{j-1}(\lambda_j - i\zeta/2) \sinh^{m-j}(\lambda_j + i\zeta/2)}{\cosh^{m}{\zeta} \lambda_j}
$$

$\leadsto$ dependence on $m$ ?

(1)+(2) $\Rightarrow$ difficult to analyse! $\Rightarrow$ new tools needed!
Emptiness formation probability

Integral representation as a single elementary block but previous expression not symmetric

\[ \tau(m) = \lim_{\xi_1, \ldots, \xi_m \to -\frac{i\zeta}{2}} \frac{1}{m!} \int_{-\infty}^{\infty} d^m \lambda \prod_{a,b=1}^{m} \frac{1}{\sinh(\lambda_a - \lambda_b - i\zeta)} \times \prod_{a<b} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\xi_a - \xi_b)} \cdot Z_m(\{\lambda\}, \{\xi\}) \cdot \det_m[\rho(\lambda_j, \xi_k)] \]

where \( Z_m(\{\lambda\}, \{\xi\}) \) is the partition function of the 6-vertex model with domain wall boundary conditions (Izergin) and \( \rho(\lambda, \xi) = [-2i\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)]^{-1} \) is the inhomogeneous version of the density for the ground state (massless regime \( \Delta = \cos \zeta, h = 0 \)).
\[
Z_m(\{\lambda\}, \{\xi\}) = \prod_{a=1}^{m} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - \xi_b) \sinh(\lambda_a - \xi_b - i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta)} \cdot \frac{\det_m \left( \frac{-i \sin \zeta}{\sinh(\lambda_j - \xi_k) \sinh(\lambda_j - \xi_k - i\zeta)} \right)}{\prod_{a>b} \sinh(\xi_a - \xi_b)}
\]

**Exact computation for** \( \Delta = 1/2 \)

The determinant structure combined with the periodicity properties at \( \Delta = 1/2 \) enable us to separate the multiple integral and to compute them:

\[
\tau_{inh}(m, \{\xi_j\}) = \frac{(-1)^{m^2 - m}}{2m^2} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b - \xi_a)}{\sinh(\xi_b - \xi_a)} \prod_{a,b=1}^{m} \frac{1}{\sinh(\xi_a - \xi_b)} \cdot \det_m \left( \frac{3 \sinh \frac{\xi_j - \xi_k}{2}}{\sinh \frac{3(\xi_j - \xi_k)}{2}} \right)
\]

\[
\tau(m) = \left( \frac{1}{2} \right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k + 1)!}{(m + k)!} = \left( \frac{1}{2} \right)^{m^2} A_m
\]

\( \rightarrow \) \( A_m \) - number of alternating sign matrices

\( \rightarrow \) first exact result for \( \Delta \neq 0 \) (and proof of one of the Razumov-Stroganov conjectures)
Asymptotic Results:

* massless case ($-1 < \Delta = \cos \zeta \leq 1$)

\[
\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = \log \frac{\pi}{\zeta} + \frac{1}{2} \int_{\mathbb{R}-i0} d\omega \frac{\sinh \frac{\omega}{2} (\pi - \zeta) \cosh^2 \frac{\omega \zeta}{2}}{\omega \sinh \frac{\pi \omega}{2} \sinh \frac{\omega \zeta}{2} \cosh \omega \zeta}
\]

\[
= \begin{cases} 
-\frac{1}{2} \log 2 & \text{for } \Delta = 0 \\
\frac{3}{2} \log 3 - 3 \log 2 & \text{for } \Delta = \frac{1}{2} \\
\log \left[ \frac{\Gamma(\frac{3}{4}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})} \right] & \text{for } \Delta = 1 \text{ (XXX chain)}
\end{cases}
\]

* massive case ($\Delta = \cosh \zeta > 1$)

\[
\lim_{m \to \infty} \frac{\log \tau(m)}{m^2} = -\frac{\zeta}{2} - \sum_{n=1}^{\infty} e^{-n\zeta} \frac{\sinh(n\zeta)}{n \cosh(2n\zeta)} \xrightarrow{\zeta \to 0} \log \left[ \frac{\Gamma(\frac{3}{4}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})} \right] \quad \text{(XXX)}
\]

\[
\xrightarrow{\zeta \to +\infty} -\infty \quad \text{(Ising)}
\]
Generating function for $\sigma^z$ correlation functions

$$Q_1^\kappa = \prod_{n=1}^{m} \left( \frac{1 + \kappa}{2} + \frac{1 - \kappa}{2} \cdot \sigma_n^z \right) = \prod_{a=1}^{m} (A + \kappa D) (\xi_a) \prod_{b=1}^{m} (A + D)^{-1} (\xi_b)$$

Generating function (polynomial in $\kappa$):

$$\langle Q_1^\kappa \rangle = \frac{\langle \psi(\{\lambda\})|Q_1^\kappa |\psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\})|\psi(\{\lambda\}) \rangle},$$

where $|\psi(\{\lambda\})\rangle$ is an eigenstate of $T(\mu)$ depending on the $N$ parameters $\lambda_j$ satisfying Bethe equations. We have,

$$\frac{1}{2} \langle (1 - \sigma_1^z)(1 - \sigma_{m+1}^z) \rangle = \frac{\partial^2}{\partial \kappa^2} \langle \left( Q_1^\kappa_{m+1} - Q_1^\kappa_{1,m} - Q_2^\kappa_{2,m+1} + Q_2^\kappa_{2,m} \right) \rangle \bigg|_{\kappa=1}$$
Master equation for $\sigma^z$ correlation functions

Let the inhomogeneities $\{\xi\}$ be generic and the set $\{\lambda\}$ be an admissible off-diagonal solution of the Bethe equations (cf. Tarasov - Varchenko). Then there exists $\kappa_0 > 0$ such, that for $|\kappa| < \kappa_0$ the expectation value of the operator $Q_{1,m}^\kappa$:

$$
\langle Q_{1,m}^\kappa \rangle = \frac{1}{N!} \oint_{\Gamma\{\xi\} \cup \Gamma\{\lambda\}} \prod_{j=1}^N \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^N \sinh^2(\lambda_a - z_b) \cdot \prod_{a=1}^m \frac{\tau_\kappa(\xi_a|\{z\})}{\tau(\xi_a|\{\lambda\})} \\
\times \frac{\det_N \left( \frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial z_k} \right) \cdot \det_N \left( \frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j} \right)}{\prod_{a=1}^N \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N \left( \frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j} \right)}.
$$

The integration contour is such that the only singularities of the integrand within the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral are the points $\{\xi\}$ and $\{\lambda\}$ (hep-th/0406190).
Twisted transfer matrix: $T_\kappa(\lambda) = A(\lambda) + \kappa D(\lambda)$, \quad $[T_\kappa(\lambda), T_\kappa(\mu)] = 0$

Eigenstates of $T_\kappa(\mu)$ and $H$ obtained from: $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$, $\{\lambda_k\}$ solution of the (twisted) Bethe equations:

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \quad j = 1, \ldots, N.$$ 

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu + \eta) + \kappa d(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu - \eta)$$

Eigenvalue $\tau_\kappa(\mu|\{\lambda\})$ of the operator $T_\kappa(\mu)$:

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa d(\mu) \prod_{k=1}^N \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}$$
Time-dependent master equation

\[ \langle Q_\kappa(m, t) \rangle = \frac{1}{N!} \int_{\Gamma\{\pm \frac{\eta}{2}\} \cup \Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{b=1}^{N} e^{it(E(z_b)-E(\lambda_b)) + im(p(z_b)-p(\lambda_b))} \times \prod_{a,b=1}^{N} \sinh^2(\lambda_a - z_b) \cdot \frac{\det_N \left( \frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial \omega_k} \right) \cdot \det_N \left( \frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j} \right)}{\prod_{a=1}^{N} \psi_\kappa(z_a|\{z\}) \cdot \det_N \left( \frac{\partial \psi(\lambda_k|\{\lambda\})}{\partial \lambda_j} \right)} \]

\[ E(z) = \frac{2 \sinh^2 \eta}{\sinh(z - \frac{\eta}{2}) \sinh(z + \frac{\eta}{2})} \]

\[ p(\lambda) = i \log \left( \frac{\sinh(\lambda - \frac{\eta}{2})}{\sinh(\lambda + \frac{\eta}{2})} \right) \]
Generating function at $\Delta = \frac{1}{2}$

Inhomogeneous case (multiple integrals can be separated):

$$\langle Q_\kappa(m) \rangle = \frac{3^m}{2^{m^2}} \prod_{a>b} \frac{\sinh 3(\xi_a - \xi_b)}{\sinh^3(\xi_a - \xi_b)} \sum_{n=0}^{m} \kappa^{m-n} \sum_{\{\xi\} = \{\xi_{\gamma_+}\} \cup \{\xi_{\gamma_-}\}} \det_{m} \Phi^{(n)}$$

$$\times \prod_{a \in \gamma_+} \prod_{b \in \gamma_-} \frac{\sinh(\xi_b - \xi_a - \frac{i\pi}{3}) \sinh(\xi_a - \xi_b)}{\sinh^2(\xi_b - \xi_a + \frac{i\pi}{3})},$$

$$\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\}, \{\xi_{\gamma_-}\}) = \begin{pmatrix}
\Phi(\xi_j - \xi_k) & \Phi(\xi_j - \xi_k - \frac{i\pi}{3}) \\
\Phi(\xi_j - \xi_k + \frac{i\pi}{3}) & \Phi(\xi_j - \xi_k)
\end{pmatrix}, \quad \Phi(x) = \frac{\sinh \frac{x}{2}}{\sinh \frac{3x}{2}}.$$
Explicit results at $\Delta = \frac{1}{2}$

If the lattice distance $m$ is not too large, the representations can be successfully used to compute $\langle Q_\kappa(m) \rangle$ explicitly.

First results for $P_m(\kappa) = 2^m \langle Q_\kappa(m) \rangle$ up to $m = 9$:

- $P_1(\kappa) = 1 + \kappa$,
- $P_2(\kappa) = 2 + 12\kappa + 2\kappa^2$,
- $P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3$,
- $P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4$,
- $P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5$,
- $P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 + 96289380\kappa^5 + 7436\kappa^6$. 
Exact vs asymptotic results at $\Delta = \frac{1}{2}$

<table>
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<th>m</th>
<th>$\langle \sigma^z_1 \sigma^z_{m+1} \rangle$ Exact</th>
<th>$\langle \sigma^z_1 \sigma^z_{m+1} \rangle$ Asymptotics</th>
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Some open problems...

- Asymptotic behavior of correlation functions: challenging the conformal limit from the lattice models

- Continuum (Field theory) models (NLS, ShG,...):
  * Approach from the lattice
  * Inverse problem for infinite dimensional representations
  * Link to Q operator and SOV methods

- Even more "sophisticated" models:
  * Hubbard: needs extended Yang-Baxter and ABA or FBA understanding
  * $\sigma$ models: again extended Yang-Baxter structures are needed