Graph Theory I

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$G$ is **simple** if each edge in $E$ is associated to a different pair of distinct vertices of $V$. In this case we can consider $E$ to be a family of size two subsets of $V$.

$G$ is **finite** if $V$ and $E$ are both finite. All graphs considered in these talks are assumed to be finite.
Let $G = (V, E)$ be a graph, $e \in E$ and let $v, w \in V$ be the vertices incident to $e$. 

$G$ delete $e$ is the graph $G - e$ obtained from $G$ by deleting $e$ from $E$. 

$G$ delete $v$ is the graph $G - v$ obtained from $G$ by deleting $v$ from $V$ and all edges incident to $v$ from $E$. 

$G$ contract $e$ is the graph $G/e$ obtained by deleting $e$ from $E$, deleting $v, w$ from $V$ and then adding a new vertex $z$ which is incident to all edges in $E - e$ which were incident to $v$ or $w$. 

Note that if $e$ is a loop then $G - e = G/e$. 
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- Note that if $e$ is a loop then $G - e = G/e$. 
A **subgraph** of $G$ is any graph which can be obtained by recursively deleting edges and isolated vertices from $G$. 

A **contraction** of $G$ is any graph which can be obtained by recursively contracting edges in $G$.

A **minor** of $G$ is any graph which can be obtained by recursively deleting or contracting edges and deleting isolated vertices from $G$. 

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Robertson-Seymour Theorem

Theorem (Robertson and Seymour)

Let $G_1, G_2, \ldots$ be an infinite sequence of graphs. Then $G_i$ is a minor of $G_j$ for some $i < j$. 

Corollary

Let $F$ be a minor closed family of graphs. Then there exists a finite set of graphs $H$ such that, for any graph $G$, $G \in F$ if and only if $G$ does not have a minor in $H$. 

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Corollary

Let $\mathcal{F}$ be a minor closed family of graphs. Then there exists a finite set of graphs $\mathcal{H}$ such that, for any graph $G$, $G \in \mathcal{F}$ if and only if $G$ does not have a minor in $\mathcal{H}$. 
Let $G$ and $H$ be graphs.

- If $H$ can be obtained from $G$ by deleting an edge belonging to a cycle of length two then $H$ is a parallel reduction of $G$, and $G$ is a parallel extension of $H$. 
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- If $H$ can be obtained from $G$ by deleting an edge belonging to a cycle of length two then $H$ is a **parallel reduction** of $G$, and $G$ is a **parallel extension** of $H$.

- If $H$ can be obtained from $G$ by contracting an edge incident to a vertex of degree two then $H$ is a **series reduction** of $G$, and $G$ is a **series extension** of $H$. If $G$ can be obtained from $H$ by a sequence of series extensions, then $G$ is a **subdivision** of $H$. 
Let $G = (V, E)$ be a graph and $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be subgraphs of $G$. 
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- The **union** of $H_1$ and $H_2$ is the subgraph of $G$ given by $H_1 \cup H_2 = (V_1 \cup V_2, E_1 \cup E_2)$. 
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- The **union** of $H_1$ and $H_2$ is the subgraph of $G$ given by $H_1 \cup H_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

- The **intersection** of $H_1$ and $H_2$ is the subgraph of $G$ given by $H_1 \cap H_2 = (V_1 \cap V_2, E_1 \cap E_2)$. 
Let $G$ be a graph.

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**Lemma**

(a) $G$ is a forest if and only if $G$ does not have the cycle of length one as a minor.
(b) $G$ is series parallel if and only if $G$ is a 2-connected loopless graph which does not have the complete graph on four vertices $K_4$ as a minor.
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**Theorem**

- $G$ is outerplanar if and only if $G$ does not have the complete graph $K_4$ or the complete bipartite graph $K_{2,3}$ as a minor.
- $G$ is planar if and only if $G$ does not have $K_5$ or $K_{3,3}$ as a minor. (Kuratowski)
- For each surface $S$, there exists a finite set of graphs $\mathcal{H}$ such that, for any graph $G$, $G$ can be embedded in $S$ if and only if $G$ has no minor in $\mathcal{H}$. (Robertson and Seymour)
Let $G = (V, E)$ be a graph and $U \subseteq V$.

- The subgraph of $G$ **induced** by $U$ is the subgraph $G[U] = G - (V - U)$. 

**Beineke's Theorem**

There exists a set $F$ of nine graphs on four, five, and six vertices such that a graph $G$ is a line graph if and only if $G$ does not have an induced subgraph in $F$. 

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- The subgraph of $G$ **induced** by $U$ is the subgraph $G[U] = G - (V - U)$.

- $G$ is a **line graph** if $G$ is simple and there exists a simple graph $H$ such that $V(G) = E(H)$ and two vertices $x, y \in V(G)$ are adjacent in $G$ if and only if $x, y$ are adjacent as edges in $H$.

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**Beineke’s Theorem**

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A graph $G$ is **claw-free** if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph.
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**Note**

$K_{1,3}$ belongs to Beineke’s list of nine excluded induced subgraphs for line graphs, so all line graphs are claw free. Many properties of line graphs extend to the larger family of claw-free graphs.
Let $G = (V, E)$ be a connected graph, $U \subseteq V$, and $k \geq 1$ be an integer.

- The set of edges of $G$ between $U$ and $V - U$ is an **edge-cut** of $G$. We denote this edge-cut by $E_G(U, V - U)$. 

$G$ is $k$-edge-connected if every edge-cut of $G$ contains at least $k$ edges. 

$U$ is a **vertex-cut** of $G$ if $G - U$ is disconnected. 

$G$ is $k$-connected if $G$ has at least $k + 1$ vertices and every vertex-cut of $G$ contains at least $k$ vertices.
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Let $G$ be a graph and $u, v$ be vertices of $G$.

- The minimum size of an edge-cut which separates $u$ and $v$ in $G$ is equal to the maximum number of pairwise edge-disjoint $uv$-paths in $G$.

- If $u$ and $v$ are not adjacent, then the minimum size of a vertex-cut which separates $u$ and $v$ in $G$ is equal to the maximum number of pairwise openly-disjoint $uv$-paths in $G$. 

**Corollary**

$G$ is $k$-edge-connected if and only if every pair of vertices of $G$ are joined by $k$ pairwise edge-disjoint paths.

$G$ is $k$-connected if and only if $G$ has at least $k+1$ vertices and every pair of non-adjacent vertices of $G$ are joined by $k$ pairwise openly-disjoint $uv$-paths in $G$. 

Menger’s Theorem

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Corollary

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- $G$ is $k$-connected if and only if $G$ has at least $k + 1$ vertices and every pair of non-adjacent vertices of $G$ are joined by $k$ pairwise openly-disjoint paths.
Theorem (Gomory and Hu)

Let $G = (V, E)$ be a connected graph. Then there exists a tree $T = (V, F)$ and a map $c : F \to \mathbb{N}$ such that, for all $u, v \in V$, the minimum size of an edge-cut which separates $u$ and $v$ in $G$ is equal to the $\min \{c(e)\}$ over all edges $e$ in the unique $uv$-path in $T$. Furthermore, if $e$ is an edge of $T$ for which this minimum is attained, and $\{U, V - U\}$ is the partition of $V$ given by the connected components of $T - e$, then $E_G(U, V - U)$ is an edge-cut in $G$ of size $c(e)$.

Someone asked during my talk if it is always possible to find a Gomory-Hu tree $T$ for $G$ such that $T$ is a spanning tree of $G$. Rob Waters pointed that $G = K_{2,3}$ is a counterexample: we cannot even find a spanning tree $T = (V, F)$ of $K_{2,3}$ and a map $c : F \to \mathbb{N}$ such that, for all $u, v \in V$, the minimum size of an edge-cut which separates $u$ and $v$ in $K_{2,3}$ is equal to the $\min \{c(e)\}$ over all edges $e$ in the unique $uv$-path in $T$. 

Let $G = (V, E)$ be a connected graph.

- A **cut-vertex** of $G$ is a vertex $v$ such that $G - v$ is disconnected i.e. a vertex-cut of size one.
Decomposing Connected Graphs: The Block/Cut-Vertex Tree

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- $G$ is **non-separable** if $G$ has no cut-vertices. (Thus $G$ is non-separable if and only if $G$ is a complete graph on at most two vertices or $G$ is 2-connected.)
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- A **block** of \( G \) is a maximal non-separable subgraph of \( G \).
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- $G$ is **non-separable** if $G$ has no cut-vertices. (Thus $G$ is non-separable if and only if $G$ is a complete graph on at most two vertices or $G$ is 2-connected.)
- A **block** of $G$ is a maximal non-separable subgraph of $G$.

**Theorem**

Let $G$ be a connected graph. Let $B$ be the set of blocks of $G$ and $X$ be the set of cut vertices of $G$. Let $T = (U, F)$ be the simple graph such that $U = B \cup X$, and $bx \in F$ if and only if $b \in B$, $x \in X$, and $x$ is a vertex of $b$. Then $T$ is a tree.
Let $G$ be a 2-connected graph.

Let $x$ and $y$ be vertices of $G$ such that $G - \{x, y\}$ has components $G_1, G_2, \ldots, G_r$, with $r \geq 2$ if $xy$ is not a multiple edge of $G$. 
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An $\{x, y\}$-component of $G$ is a subgraph of $G$ induced by $V(G_i) \cup \{x, y\}$, but with any edges joining $x$ and $y$ deleted. In addition, if $xy \in E(G)$, then the subgraph induced by $\{x, y\}$ is a trivial $\{x, y\}$-component of $G$. 

A hinge $\{x, y\}$ of $G$ is Type I if $G$ has exactly two $\{x, y\}$-components and Type II otherwise.
Decomposing 2-Connected Graphs: Hinges

- Let $G$ be a 2-connected graph.
- Let $x$ and $y$ be vertices of $G$ such that $G - \{x, y\}$ has components $G_1, G_2, \ldots, G_r$, with $r \geq 2$ if $xy$ is not a multiple edge of $G$.
- An $\{x, y\}$-component of $G$ is a subgraph of $G$ induced by $V(G_i) \cup \{x, y\}$, but with any edges joining $x$ and $y$ deleted. In addition, if $xy \in E(G)$, then the subgraph induced by $\{x, y\}$ is a trivial $\{x, y\}$-component of $G$.
- Let $H$ be an $\{x, y\}$-component of $G$ and put $H' = G - (H - \{x, y\})$. $H$ is excisable if $H$ is not trivial and either $H$ or $H'$ is a 2-connected graph or a multiple edge.
Let $G$ be a 2-connected graph.

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If an excisable $\{x, y\}$-component $H$ of $G$ exists, we say that $\{x, y\}$ is a hinge of $G$ and $H$ is a hinge component of $G$. 
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A hinge $\{x, y\}$ of $G$ is Type I if $G$ has exactly two $\{x, y\}$-components and Type II otherwise.
Construct the **augmented graph** $G^{aug}$ from $G$ by adding a new edge incident to $x$ and $y$ for each hinge $\{x, y\}$ and each excisable $\{x, y\}$-component of $G$. These new edges are called **virtual edges**. Two distinct hinge components of $G$ give rise to the same virtual edge if and only if they are the two $\{x, y\}$-components of the same hinge $\{x, y\}$ of Type I.
Composing 2-Connected Graphs: Cleavage Units

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- If $H$ is an excisable $\{x, y\}$-component of $G$, the two graphs $D_1$ and $D_2$ derived from $H$ and $H'$ by adjoining to each of $H$ and $H'$ the virtual edge $e$ associated with $H$ are called the **cleavage graphs** of $G$ at $e$. 

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Construct the **augmented graph** \( G^{\text{aug}} \) from \( G \) by adding a new edge incident to \( x \) and \( y \) for each hinge \( \{x, y\} \) and each excisable \( \{x, y\} \)-component of \( G \). These new edges are called **virtual edges**. Two distinct hinge components of \( G \) give rise to the same virtual edge if and only if they are the two \( \{x, y\} \)-components of the same hinge \( \{x, y\} \) of Type I.

If \( H \) is an excisable \( \{x, y\} \)-component of \( G \), the two graphs \( D_1 \) and \( D_2 \) derived from \( H \) and \( H' \) by adjoining to each of \( H \) and \( H' \) the virtual edge \( e \) associated with \( H \) are called the **cleavage graphs** of \( G \) at \( e \).

The **cleavage units** of \( G \) are the minimal cleavage graphs obtained by recursively constructing cleavage graphs from cleavage graphs. (No cleavage unit of \( G \) can have a hinge, and each virtual edge of \( G \) belongs to exactly two cleavage units.)
The cleavage unit tree $T$ of $G$ is the graph whose vertices and edges are the cleavage units and virtual edges, respectively, of $G$, in which a cleavage unit $D$ and a virtual edge $e$ are incident in $T$ if and only if $e$ is an edge of $D$. 

Tutte showed that the cleavage unit tree of a 2-connected graph $G$ is indeed a tree and that each cleavage unit of $G$ is either a 3-connected simple graph or a cycle of length at least three, or a multiple edge with at least three edges.
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