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Newton Institute, 15 January, 2008
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A graph $G$ is **bipartite** if its vertices can be partitioned into two sets $X, Y$ such that each edge of $G$ is incident with a vertex in $X$ and a vertex in $Y$. 

**Theorem**

Let $G$ be a bipartite graph with bipartition $\{X, Y\}$. $G$ has a matching which saturates $X$ if and only if all $S \subseteq X$ have at least $|S|$ neighbours in $Y$. (Hall)

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Let $G = (V, E)$ be a graph.

- $G$ has a perfect matching if and only if $k^{odd}(G - S) \leq |S|$ for all $S \subseteq V$. (Tutte)
- The maximum size of a matching in $G$ is equal to $\min\{(|V| - k^{odd}(G - S) + |S|)/2\}$ over all $S \subseteq V$. (Berge)
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**Note**

Edmonds gave a polynomial time algorithm for determining a maximum size matching in a graph.
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- We can determine a maximum size independent set of vertices in a line graph since this is equivalent to determining a maximum size matching in the original graph (which we can do by Edmond’s algorithm).
- Minty showed that Edmond’s algorithm could also be used to give a polynomial time algorithm for finding a maximum size independent set of vertices for the more general family of claw-free graphs. (An error in Minty’s algorithm was subsequently corrected by Nakamura and Tamura.)
Let $G = (V, E)$ be a graph and $k \geq 1$ be an integer.

- A **proper $k$-vertex-colouring** of $G$ is an assignment of $k$ colours to the vertices of $G$ such that no pair of adjacent vertices receive the same colour. (Equivalently, it is a partition of $V$ into $k$ independent sets of vertices.)
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**Theorem (Brooks)**

Let \( G \) be a connected simple graph of maximum degree \( \Delta \). Then \( \chi(G) \leq \Delta + 1 \) with equality if and only if \( G = K_{\Delta+1} \), or \( \Delta = 2 \) and \( G \) is an odd cycle.
Map Colour Theorems

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- If $G$ can be embedded in a surface of Euler characteristic $c \neq 2$ then $\chi(G) \leq \lfloor (7 + \sqrt{49 - 24c})/2 \rfloor$. (Heawood)

This bound is best possible for all surfaces except the Klein bottle. (Ringel and Youngs)
Perfect Graphs

Let $G = (V, E)$ be a simple graph and $U \subseteq V$.

- $U$ is a **clique** of $G$ if the subgraph of $G$ induced by $U$ is a complete graph. The **clique number** of $G$, $\omega(G)$, is the maximum size of a clique in $G$.
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- $G$ is **perfect** if $\chi(H) = \omega(H)$ for all induced subgraphs $H$ of $G$. 

An odd cycle of length at least five and its complement are examples of minimal non-perfect graphs. Berge conjectured that they are the only such examples.

**Theorem (Chudnovsky, Robertson, Seymour and Thomas)**

Let $G$ be a simple graph. Then $G$ is perfect if and only if $G$ does not contain an odd cycle of length at least five or its complement as an induced subgraph.
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- The **chromatic index** of $G$, $\chi'(G)$, is the minimum value of $k$ such that $G$ has a proper $k$-edge-colouring. Clearly $\chi'(G) \geq \Delta$. 
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- For each $k \geq 3$, it is NP-complete to decide if a graph has a proper $k$-edge-colouring. (Holyer)
The Shannon-Vizing Theorem

Let $G$ be a graph of maximum degree $\Delta$.

- $\chi'(G) \leq 3\Delta/2$. (Shannon)
- $\chi'(G) \leq \Delta + \mu(G)$ where $\mu(G)$ denotes the maximum multiplicity of an edge of $G$. (Vizing)
- If $G$ is bipartite then $\chi'(G) = \Delta$. (König)
Let \( G = (V, E) \) be a graph and \( \Gamma \) be an additive abelian group.

- Construct a digraph \( \vec{G} \) by giving the edges of \( G \) an arbitrary orientation. For \( U \subseteq V \) and \( \bar{U} = V - U \), let \( E^+(U) \) be the set of arcs from \( U \) to \( \bar{U} \) in \( \vec{G} \) and \( E^-(U) = E^+(\bar{U}) \).
Let $G = (V, E)$ be a graph and $\Gamma$ be an additive abelian group.

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- Let $f : E(\vec{G}) \rightarrow \Gamma$ and put $f^+(U) = \sum_{e \in E^+(U)} f(e)$ and $f^-(U) = \sum_{e \in E^-(U)} f(e)$.

A $\Gamma$-flow $f$ for $G$ with respect to $\vec{G}$ is nowhere-zero if $f(e) \neq 0$ for all $e \in E(G)$.
Let $G = (V, E)$ be a graph and $\Gamma$ be an additive abelian group.

- Construct a digraph $\tilde{G}$ by giving the edges of $G$ an arbitrary orientation. For $U \subseteq V$ and $\bar{U} = V - U$, let $E^+(U)$ be the set of arcs from $U$ to $\bar{U}$ in $\tilde{G}$ and $E^-(U) = E^+(\bar{U})$.

- Let $f : E(\tilde{G}) \to \Gamma$ and put $f^+(U) = \sum_{e \in E^+(U)} f(e)$ and $f^-(U) = \sum_{e \in E^-(U)} f(e)$.

- $f$ is a $\Gamma$-flow for $G$, with respect to $\tilde{G}$, if $f^+(v) = f^-(v)$ for all $v \in V(G)$. 


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- $f$ is a $\Gamma$-flow for $G$, with respect to $\vec{G}$, if $f^+(v) = f^-(v)$ for all $v \in V(G)$.

- If, in addition, $f(e) \neq 0$ for all $e \in E(G)$, then $f$ is a nowhere-zero $\Gamma$-flow for $G$. 

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Graph Theory II
The condition $f^+(v) = f^-(v)$ for all $v \in V(G)$ is equivalent to the apparently stronger condition that $f^+(U) = f^-(U)$ for all $U \subseteq V(G)$. This implies that, if $G$ has a nowhere-zero $\Gamma$-flow, then $G$ is bridgeless. (A bridge in $G$ is an edge-cut of size one.) Since reversing the orientation on an edge $e$ of $\vec{G}$ is equivalent to replacing $f(e)$ by $-f(e)$, the number of distinct nowhere-zero $\Gamma$-flows for $G$ is independent of the chosen orientation $\vec{G}$ of $G$. 

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Graph Theory II
Group valued flows, continued

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- A nowhere-zero $k$-flow for $G$ is a nowhere-zero $\mathbb{Z}$-flow, $f$, such that $|f(e)| \leq k - 1$ for all $e \in E(G)$. 

G has a nowhere-zero $k$-flow if and only if $G$ has a nowhere-zero $\mathbb{Z}_k$-flow (Tutte).

The number of distinct nowhere-zero $\Gamma$-flows for $G$ is the same for all abelian groups $\Gamma$ of the same order. (Tutte)

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Conjecture (Tutte)

Let $G$ be a bridgeless graph.

- $G$ has a nowhere zero 5-flow.
- If $G$ has no edge-cuts of size three then $G$ has a nowhere zero 3-flow.
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Let \( G = (V, E) \) be a graph. For each positive integer \( t \), let \( P_G(t) \) be the number of proper \( t \)-vertex-colourings of \( G \). (By definition \( P_G(t) \equiv 1 \) if \( E = \emptyset \), and \( P_G(t) \equiv 0 \) if \( G \) has a loop.)
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**Deletion-Contraction Lemma**

Let $G$ be a graph and $e$ be an edge of $G$ which is not a loop. Then

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t).$$
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This implies that $P_G(t)$ is a polynomial in $t$, the **chromatic polynomial** of $G$. 


Let $G = (V, E)$ be a graph. For each positive integer $t$, let $F_G(t)$ be the number of nowhere-zero $\mathbb{Z}_t$-flows of $G$. (By definition $F_G(t) \equiv 1$ if $E = \emptyset$.)
The Flow Polynomial

Let $G = (V, E)$ be a graph. For each positive integer $t$, let $F_G(t)$ be the number of nowhere-zero $\mathbb{Z}_t$-flows of $G$. (By definition $F_G(t) \equiv 1$ if $E = \emptyset$.)

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**Lemma (Tutte)**

If $G$ is a connected plane graph and $G^*$ is its planar dual then

$$tF_G(t) = P_{G^*}(t).$$
Let $G = (V, E)$ be a graph.

- For $A \subseteq E$, let $k(A)$ be the number of connected components in the subgraph $(V, A)$.
The Tutte Polynomial

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- The **rank** of $A$ is $r(A) = |V| - k(A)$.
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- For \( A \subseteq E \), let \( k(A) \) be the number of connected components in the subgraph \((V, A)\).
- The rank of \( A \) is \( r(A) = |V| - k(A) \).
- The Tutte polynomial of \( G \) is the 2-variable polynomial given by

\[
T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.
\]
The Tutte Polynomial

Let $G = (V, E)$ be a graph.

- For $A \subseteq E$, let $k(A)$ be the number of connected components in the subgraph $(V, A)$.
- The rank of $A$ is $r(A) = |V| - k(A)$.
- The Tutte polynomial of $G$ is the 2-variable polynomial given by

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$ 

Deletion-Contraction Lemma

Let $G$ be a graph and $e$ be an edge of $G$. Then

- $T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y)$ if $e$ is neither a loop not a bridge,
- $T_G(x, y) = xT_{G-e}(x, y)$ if $e$ is a bridge,
- $T_G(x, y) = yT_{G-e}(x, y)$ if $e$ is a loop.
Let $G = (V, E)$ be a graph.

- If $G$ is embedded in the plane and $G^*$ is its planar dual then $T_G(x, y) = T_{G^*}(y, x)$. 

$P_G(t) = (-1)^r(E) t^k(E) T_G(1-t, 0)$

$F_G(t) = (-1)^{|V|} T_G(0, 1-t)$

$T_G(1, 1)$ is the number of spanning trees of $G$.

$T_G(2, 0)$ is the number of acyclic orientations of $G$.

$T_G(0, 2)$ is the number of totally-cyclic orientations of $G$. 

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Graph Theory II
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$P_G(t)$ and $F_G(t)$ are the Poincaré polynomial and the logarithmic Frobenius polynomial respectively.
Let $G = (V, E)$ be a graph.

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The Pott’s model partition function, or multivariate Tutte polynomial, of a graph $G = (V, E)$ is the $(|E| + 1)$-variable polynomial given by

$$Z_G(q, w) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e,$$

where $w = (w_e)_{e \in E}$ is a vector of indeterminates.
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**Deletion-Contraction Lemma**

Let \( G \) be a graph and \( e \) be an edge of \( G \). Then

\[
Z_G(q, w) = Z_{G-e}(q, w|_{E-e}) + w_e Z_{G/e}(q, w|_{E-e}).
\]
The **Pott’s model partition function**, or **multivariate Tutte polynomial**, of a graph $G = (V, E)$ is the $(|E| + 1)$-variable polynomial given by

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**Lemma**

$$T_G(x, y) = (x - 1)^{-k(E)}(y - 1)^{-|V|}Z_G((x - 1)(y - 1), (y - 1)1).$$
Theorem (Fortuin-Kasteleyn)

Let $G$ be a graph and $q$ be a positive integer. Let $S = \{1, 2, \ldots, q\}$. Then

$$Z_G(q, w) = \sum_{\sigma: V \rightarrow S} \prod_{e \in E} (1 + w_e \delta_e),$$

where $\delta_e = 1$ if $\sigma$ maps the end-vertices of $e$ onto the same element of $S$, and $\delta_e = 0$ otherwise.