Connectivity:

\[(A, B) \text{ partition of } E.\]

\[\lambda(A, B) = \gamma(A) + \gamma(B) - \gamma(M).\]

What does this measure?

- Such \( \lambda(A, B) = 0 \) 
  A separation

- \( \lambda(A, B) = 1 \) 
  A 2-separation

- \( \lambda(A, B) = 2 \) 
  A 3-separation

\( \gamma(M) \) is not necessarily in \( E \).
Matroid connectivity generalises vertex connectivity.

\[ G = (V, E), \ (A, B) \text{ partition of } E. \]

- Assume \( A, B \) induce connected subgraphs:
  \[ r_{m(G)}(A) = |V(A)| - 1 \]
  \[ r_{m(G)}(B) = |V(B)| - 1 \]
  \[ r_{m(G)}(G) = |V(G)| - 1 \]

\[ \lambda_{m(G)}(A, B) = |V(A) \cup V(B)| - 1 \]

\( \lambda_{m(G)}(A, B) \) is the size of a vertex cut.

\[ |V(A) \cup V(B)| = 2 \]
Menger's Theorem for Matroids

\[ \kappa(A, B) = \min \{ \lambda(A', B') : A' \supset A, B' \supset B \} \]

\( \kappa(A, B) \) is an upper bound for the amount of communication between A and B.

Tutte's Linking Lemma:
There exists partition \( X, Y \) of \( E - (A \cup B) \) s.t.

\[ \lambda_{\text{min}}(X, Y)(A, B) = \kappa_m(A, B). \]

Easy proof.
\[ M \text{ k-connected if no non-trivial } k\text{-separation.} \]

\[ G_1 \quad G_2 \]

\[ M(G), M(G) \text{ not connected.} \]

\[ \bullet G \text{ 2-connected } \iff M(G) \text{ 2-connected.} \]

\[ \bullet G \text{ 3-connected } \iff M(G) \text{ 3-connected.} \]

Almost - up to parallel edges.
Bill's decomposition of 2-connected graphs into 3-connected pieces extends to matroids.

⇐ Often suffices to solve problems for 3-connected matroids.
**Regular Matroids**

A matrix over \( \mathbb{Q} \) is **totally unimodular** if all subdeterminants are in \( \{0, \pm 1\} \).

Matroid \( M \) is **regular** if it can be represented by a totally unimodular matrix.

**Examples:**
- **Graphic Matroids**
- **CoGraphic Matroids**

\[
R_{10}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Theorem (Tutte)

The following are equivalent.

(1) M is regular
(2) M is representable over all fields
(2) M is representable over GF(2) and if (not characteristic 2).
(3) M has no minor isomorphic to

\( U_{2,4}, F_7, F_7^x \).

Beautiful!

But of limited use algorithmically.
Direct Sums

2-sums → Preserve Regularity
3-sums

Theorem (Seymour):

A regular 1ff can be built by direct sums, 2-sums and 3-sums from graphic matroids, cographic matroids and copies of $R_{10}$.

Leads to polynomial time recognition algorithm
Oracle Complexity

Determining if $M$ is binary is provably exponential. The binary spike.
Ternary Classes:

**Dyadic** - matrices over \( \mathbb{Q} \), subdeterminants so, \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \).

\[ \text{Dyadic} \]

\[ = \mathbb{GF}(3) \setminus \mathbb{GF}(5) \]

\[ = \mathbb{GF}(3) \setminus \mathbb{Q} \]

Excluded Minors? Structure?
$\mathfrak{U}$ - matroids

(should be complex unimodular)

matrices over $\mathbb{C}$ subdeterminants

modulo $1$.

$\mathfrak{U}$ - matroids

$= \mathbb{GF}(3) \cap \mathbb{GF}(4)$

Excluded minors $\checkmark$

Structure $?$
Golden Mean.

\( \lambda, \beta \) roots of \( x^2 - x - 1 = 0 \).

Matrices over \( \mathbb{R} \), subdeterminants

So, \( \lambda^i \beta^j \).

Vertical (unpublished)

Golden mean

\[ = \text{GF}(4) \cap \text{GF}(5) \]
Chromatic Polynomials of Matroids $P(M; \lambda)$

$G$ a graph $P(M(G); \lambda) = P(G; \lambda)$

(essentially)

$P(M^x(G); \lambda) =$ Flow Polynomial of $G$

Theorem: $M$, $GF(q)$ - represented by $A$. Then,

$$\max \{ k \text{ s.t. } \exists \text{ a subspace } W \text{ of } PG(\mathbb{F}, q) \text{ s.t. } W \cap A = \phi \text{ and } r(W) = r - k \}$$

$$= \min \{ k \text{ s.t. } P(M; q^k) > 0 \}$$

$\Rightarrow$ CRAMP $\Rightarrow$ ROTA

The critical problem