Chromatic Factorisation of Graphs

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Joint work with Graham Farr
Overview

- Chromatic Polynomial
- Basic Properties
- Chromatic Factorisation
- Certificate of Factorisation
- Infinite Family of Graphs that have Chromatic Factorisations
The Chromatic Polynomial

$P(G, \lambda)$ gives the number of proper $\lambda$-colourings of a graph $G$

$$P(G, \lambda) = \lambda^5 - 8\lambda^4 + 24\lambda^3 - 31\lambda^2 + 14\lambda$$

$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$$

- $P(G, 0) = 0$
- $P(G, 1) = 0$
- $P(G, 2) = 0$
- $P(G, 3) = 6$
- $P(G, 4) = 72$
- $\chi(G) = 3$
Addition-identification

\[ P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda) \]
Deletion-contraction

\[ P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda) \]
Chromatic polynomial of graph with more than a single component

\[ P(G, \lambda) = P(H_1, \lambda)P(H_2, \lambda) \]
Clique-gluings

$G$ is an $r$-gluing, or clique-gluing, of graphs $H_1$ and $H_2$, if $G$ can be obtained by identifying an $r$-clique in $H_1$ with an $r$-clique in $H_2$. 
A graph is *clique-separable* if it is isomorphic to the graph obtained by an $r$-gluing of graphs $H_1$ and $H_2$.

\[ P(G, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)} \]
Chromatic Equivalence

- $G$ is chromatically equivalent to $H$ if $P(G, \lambda) = P(H, \lambda)$
- $G \sim H$
Motivation

- Large amount of research on roots of chromatic polynomials
- Little research into the algebraic theory of chromatic roots
- First step in finding roots of a polynomial is factorisation
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- First step in finding roots of a polynomial is factorisation
If there exist graphs $G, H_1, H_2$ such that

$$P(G, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)}$$

where $r \leq \min\{\chi(H_1), \chi(H_2)\}$, then $P(G, \lambda)$ has a \textit{chromatic factorisation} with \textit{chromatic factors} $P(H_1, \lambda)$ and $P(H_2, \lambda)$.

If either $H_1$ or $H_2$ is the complete graph $K_s$ then $r < s$. 
$P(G, \lambda)$ has a chromatic factorisation:

- $G$ is an $r$-gluing of graphs $H_1$ and $H_2$,
- $G \sim H$, and $H$ is an $r$-gluing of graphs $H_1$ and $H_2$
- Any others?
$P(G, \lambda)$ has a chromatic factorisation
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- $G \sim H$, and $H$ is an $r$-gluing of graphs $H_1$ and $H_2$
- Any others?
Does there exist $P(G, \lambda)$ that has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph?

Yes ...

<table>
<thead>
<tr>
<th>$n$</th>
<th>Chromatic polynomials</th>
<th>Non-isomorphic graphs</th>
</tr>
</thead>
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<tr>
<td>8</td>
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<td>25</td>
<td>97</td>
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<tr>
<td>10</td>
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<td>3018</td>
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<tr>
<td>$n \leq 10$</td>
<td>512</td>
<td>3118</td>
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Certificate of Factorisation

Sequence of steps $P_0, \ldots, P_i$

- $P_0$ is $P(G, \lambda)$
- $P_i$ is $\frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)}$
- $P_j$ is formed from $P_{j-1}$ by a certification step
A Certificate of Factorisation: Certificate 1

\[ P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda) \]  \hspace{1cm} (1)

\[ = \frac{P(H_1, \lambda)P(H_3, \lambda)}{P(K_s, \lambda)} + \frac{P(H_1, \lambda)P(H_4, \lambda)}{P(K_s, \lambda)} \]  \hspace{1cm} (2)

\[ = P(H_1, \lambda) \left( \frac{P(H_3, \lambda)}{P(K_s, \lambda)} + \frac{P(H_4, \lambda)}{P(K_t, \lambda)} \right) \]  \hspace{1cm} (3)

\[ = \frac{P(H_1, \lambda)}{P(K_r, \lambda)} \left( \frac{P(K_r, \lambda)P(H_3, \lambda)}{P(K_s, \lambda)} + \frac{P(K_r, \lambda)P(H_4, \lambda)}{P(K_t, \lambda)} \right) \]  \hspace{1cm} (4)

\[ = \frac{P(H_1, \lambda)}{P(K_r, \lambda)} (P(H_5, \lambda) + P(H_6, \lambda)) \]  \hspace{1cm} (5)

\[ = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)} \]  \hspace{1cm} (6)

(1) add-ident., (2) clique-gluing, (3) common factor,
(4) multiply by \( \frac{P(K_r, \lambda)}{P(K_r, \lambda)} \), (5) clique-gluing and (6) add-ident.
Certification Steps:

- Addition-identification
- Deletion-contraction
- Chromatic equivalence
- Clique-gluings
- Basic algebra
Certificates of Factorisation

- **Simple**
  - Graph is clique-separable
  - Graph is chromatically equivalent to a clique-separable graph

- Identified cases where $P(G, \lambda)$ has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph
  - Certificates of factorisation, $n \leq 9$

- Identified an infinite family of graphs that have a chromatic factorisation
  - Not clique-separable
  - Certificate of factorisation for this family
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A Certificate of Factorisation

- $G$ is $C_{4n+1} + (0, 2n + 1) + (2n, 4n)$, $n \geq 2$
- $H_1$ is $C_{2n+1}$

G:\[\begin{array}{c}
1 & 2 & 2n-2 & 2n-1 & 2n & 2n+1 \\
4n & 4n-1 & 2n & 2n+2 & 2n+1 & 2n+2
\end{array}\]

H_1:\[\begin{array}{c}
1 & 2 & 2n-2 & 2n-1 \\
0 & 2n & 2n+1 & 2n+2
\end{array}\]

H_2:\[\begin{array}{c}
2 & 2n-1 \\
0 & 2n+1 & 2n+2
\end{array}\]
A Certificate of Factorisation

\[ P(G, \lambda) = P(G + (0, 2n), \lambda) + P(G/(0, 2n), \lambda) \]
A Certificate of Factorisation

\[ = P(G + (0, 2n), \lambda) + P(G/(0, 2n), \lambda) \]
\[ = \frac{P(C_{2n+1}, \lambda)P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n+1}, \lambda)P(C_{2n}, \lambda)}{P(K_1, \lambda)} \]
A Certificate of Factorisation

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= \frac{P(C_{2n+1}, \lambda)P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n+1}, \lambda)P(C_{2n}, \lambda)}{P(K_1, \lambda)}
\]

\[
= P(C_{2n+1}, \lambda) \left( \frac{P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)}{P(K_1, \lambda)} \right)
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A Certificate of Factorisation

\[ A = \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} \left( \frac{P(H_3, \lambda)P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)P(K_3, \lambda)P(K_2, \lambda)}{P(K_1, \lambda)P(K_2, \lambda)} \right) \]

\[ = \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} \left( P(H_5, \lambda) + P(H_6, \lambda) \right) \]
A Certificate of Factorisation

\[ \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} \left( P(H_5, \lambda) + P(H_6, \lambda) \right) = \frac{P(C_{2n+1}, \lambda)P(H_2, \lambda)}{P(K_3, \lambda)} \]

\[
\begin{align*}
C_{2n+1} & = & \text{Cyclic graph with } 2n+1 \text{ vertices} \\
\end{align*}
\]

\[
\begin{align*}
\text{C}_2n+1 & \quad \text{with labels } 0, 1, 2, \ldots, \text{up to } 2n+1 \text{ vertices} \\
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$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda)$

$$= \frac{P(C_{2n+1}, \lambda)P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n+1}, \lambda)P(H_4, \lambda)}{P(K_1, \lambda)}$$

$$= P(C_{2n+1}, \lambda) \left( \frac{P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(H_4, \lambda)}{P(K_1, \lambda)} \right)$$

$$= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} \left( \frac{P(H_3, \lambda)P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(H_4, \lambda)P(K_3, \lambda)P(K_2, \lambda)}{P(K_1, \lambda)P(K_2, \lambda)} \right)$$

$$= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} (P(H_5, \lambda) + P(H_6, \lambda))$$

$$= \frac{P(C_{2n+1}, \lambda)P(H_2, \lambda)}{P(K_3, \lambda)}$$
Theorem

There exists an infinite family of non-clique-separable graphs $\mathcal{G}$ such that for all $G \in \mathcal{G}$, $P(G, \lambda)$ has a chromatic factorisation.

Note: Every $G \in \mathcal{G}$ satisfies Certificate 1 with $H_1 \cong C_{2n+1}$, $n \geq 2$.

Theorem

Graphs in $\mathcal{G}$ are the only graphs that have a chromatic factorisation in the form of Certificate 1 with $r = 3$ and $H_1 \cong C_{2n+1}$.
Proof (idea)
Some properties used in proof

\( P(G, \lambda) \) has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph.

Let \( t_G, t_1, t_2 \) be the number of triangles in \( G, H_1 \) and \( H_2 \) respectively. Then

- \( t_G = t_1 + t_2 - \binom{r}{3} \)
- If \( r = 3 \) then
  - \( t_1 = 0 \)
  - \( t_G = t_2 - 1 \)
- In the case where \( P(G, \lambda) \) has a chromatic factorisation in the form of Certificate 1:
  - \( t_G = 0 \)
  - \( t_2 = 1 \)
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Some properties used in proof

$P(G, \lambda)$ has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph.

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In the case where $P(G, \lambda)$ has a chromatic factorisation in the form of Certificate 1:

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Proof (idea)
Some observations

- \( H_1 \cong H_3 / uv \)
- \( H_4 \cong H_1 / uv \cong C_{2n} \)

\[ \begin{align*}
G & \quad = \quad G + uv + G \setminus uv \\
H_1 & \quad \cong \quad H_3 / uv \\
H_4 & \quad \cong \quad H_1 / uv \cong C_{2n}
\end{align*} \]
Proof (idea)

Now

- $H_1 \cong H_3/uv \cong C_{2n+1}$

Three options for $H_3$

- $C_{2n+2} + av$
- $C_{2n+2}$
- $C_{2n+2} + av + bu$

![Diagrams](a), (b), (c)
Three options for $H_3$

Certificate 1 requires:

- $H_5$ isomorphic to a $K_2$-gluing of $H_3$ and $K_3$
- $H_5 \setminus e$ isomorphic to $H_2$
- $P(H_5/ e, \lambda)$ isomorphic to $P(H_6, \lambda) = \frac{P(C_{2n}, \lambda) P(K_3, \lambda) P(K_2, \lambda)}{P(K_2, \lambda) P(K_1, \lambda)}$
$H_3$ is option (c)

So
- $H_3$ is isomorphic to $C_{2n+2} + av + bu$
- $H_5$ is isomorphic to $K_2$-gluing of $H_3$ and $K_3$ on edge $bv$

Thus
- $H_2$ is isomorphic to $H_5 \setminus bv$

and
- $G + uv$ is a $K_2$-gluing of $H_3$ and $C_{2n+1}$ on the edge $uv$
- $G$ belongs to the family of graphs
$H_3$ is option (c)

So

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\begin{figure}
\centering
\includegraphics[width=\textwidth]{graphs.png}
\caption{Graphs $H_2$, $H_3$, and $H_5$}
\end{figure}
So

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Thus

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and

- $G + uv$ is a $K_2$-gluing of $H_3$ and $C_{2n+1}$ on the edge $uv$
- $G$ belongs to the family of graphs

\[ H_3 \]

\[ G + uv \]

\[ G \]
Chromatic Factorisation of $G \in \mathcal{G}$.
Further Work

- Identify some properties of non-clique-separable graphs that have chromatic factorisations
- Identify other infinite families of non-clique-separable graphs that have a chromatic factorisation
- Devise certificates to explain different types of chromatic factorisation
- Study chromatic roots using algebraic means