On the Roots of Independence and Open Set Polynomials

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There are many polynomials that are well in graph theory:

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But polynomials are useful tools in general as generating functions for combinatorial sequences. In this way they efficiently encode a sequence of numbers as an algebraic structure. For example, a theorem due to Isaac Newton, states the following:
Theorem 1 (Newton) Suppose that $a_0, a_1, \ldots, a_n$ is a sequence of positive terms. Then the sequence is unimodal, that is, nondecreasing then nonincreasing, if the polynomial
\[ \sum a_i x^i \]
has all real roots.
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\[
\sum a_i x^i
\]
has all real roots.

(In fact, more is true – the result holds even if the roots lie in the sector of the complex plane whose argument lies between \( 2\pi/3 \) and \( 4\pi/3 \).)
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We’re going to focus on two such combinatorial sequences: generating functions for independent sets of a graph and of open sets in a finite topology. What can we say about the roots of such polynomials?
Definition 1 Let $G$ be a graph (finite and undirected). Then the independence polynomial of $G$, $i(G, x)$ is defined to be

$$i(G, x) = \sum_{k \geq 0} i_k x^k$$

where $i_k$ is the number of independent sets of size $k$ in $G$. 
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Determining the value of $i(G, z)$ for any complex $z \neq 0$ is NP-hard [BHosh].
Some examples:

- $i(K_k, x) = 1 + nx$
- $i(K_k, x) = (1 + x)^n$
- $i(C_5, x) = 1 + 5x + 5x^2$
- $i(K_{n,n}, x) = 2(1 + x)^n - 1$
Some examples:

- \( i(K_k, x) = 1 + nx \)
- \( i(\overline{K}_k, x) = (1 + x)^n \)
- \( i(C_5', x) = 1 + 5x + 5x^2 \)
- \( i(K_{n,n}, x) = 2(1 + x)^n - 1 \)
- \( i(C_n^d, x) = \sum_{k=0}^{\left\lfloor \frac{n}{d+1} \right\rfloor} \frac{n}{n - dk} \binom{n - dk}{k} x^k \) (with a musical application to counting consonant chords in an \( n \)-tet scale) [BHosh].
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Bounding the Roots of Independence Polynomials
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**Bounding the Roots of Independence Polynomials**

**Question 1** How large can the modulus of a root of an independence polynomial be for graphs of order $n$?
Theorem 2 (BN) Let $\beta \geq 2$ be fixed. Let $r_\beta(n)$ be the maximum modulus of a root of the independence polynomial of a graph of order $n$ with independence number $\beta$. Then

$$r_\beta(n) = \left( \frac{n}{\beta - 1} \right)^{\beta-1} + O(n^{\beta-2}).$$
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The result is best possible: Let $G_{l,\beta}$ be the following graph:
Each $A_i$ is a clique, with $|A_i| = l$ for $i \leq \beta - 2$ and $|A_{\beta-1}| = l + r - 1$ ($n = l(\beta - 1) + r$ where $r \in \{1, \ldots, \beta - 1\}$), and $v$ is joined to all but one vertex in each $A_j$ ($j = 1 \ldots, \beta - 1$).
Then

\[ i(G_{l,\beta}(x)) = x(1+x)^{\beta-1} + (1+lx)^{\beta-2}(1+(l+r-1)x) \]

has sign \((-1)^{\beta-1}\) at \(x = -l^{\beta-1}\), but has sign \((-1)^{\beta}\) as \(x \to -\infty\). Thus \(i(G_{l,\beta}(x))\) has a root to the left of

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\[ i(G_{l, \beta}(x)) = x(1 + x)^{\beta - 1} + (1 + lx)^{\beta - 2}(1 + (l + r - 1)x) \]

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What about special families of graphs?

**Theorem 3 (BN)** Let \(G\) be a graph with independence number \(\beta\) that is the line graph of a tree. Then all of the roots of the independence polynomial of \(G\) have modulus at most \(\binom{\beta}{2}\).
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A graph \(G\) with independence number \(\beta\) is **well covered** if all of the maximal independent sets have size \(\beta\).
Theorem 4 (BDN) All of the roots of the independence polynomial of a well covered graph $G$ with independence number $\beta$ lie in the annulus $\frac{1}{n} \leq |n| \leq \beta$, with a root on the boundary iff $G$ is complete.
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The bound of \( \beta \) is indeed best possible: We form the graph \( L^k_\beta \) on the vertex set \( \{1, \ldots, \beta\}^k \) with two \( k \)-tuples adjacent iff they agree in a coordinate. Then

- \( L^k_\beta \) has independence number \( \beta \)
- \[ i(L^k_\beta, x) = \sum_{j=0}^{\beta} \binom{\beta}{j} (j!)^{k-1} x^j \]
- All of the zeros of \( L^k_\beta \) are real and negative
- If \( 2^{k-1} \geq \beta \geq 1 \) then the smallest zero of \( L^k_\beta \) lies in the interval \( (-\beta, -\beta(1 - 2^{-k})) \).
In fact, these polynomials are related to one classical orthogonal polynomials, namely the Legendre polynomials. Also, the coefficients are unimodal (as the roots are all real), with a peak approaching $\beta - 1$. 
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**Lemma 1** Let $G$ and $H$ be graphs, and let $G[H]$ be the lexicographic product of $G$ and $H$ (a copy of $H$ is substituted for every vertex of $G$). Then

\[ i(G[H], x) = i(G, i(H, x) - 1). \]
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The closure of the roots of independence polynomials are not only of positive measure – they actually contain the whole complex plane!
Theorem 5 (BHN) The independence roots of the family $L^k_{\beta}[K_n][\overline{K_m}]$ are dense in $\mathbb{C}$.
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There is something interesting going in the lemma about the behaviour of the independence polynomial of the lexicographic product. In particular, what happens to the roots of the independence polynomial of the repeated lexicographic product $G^k$ of a graph with itself $k$ times? You’d be surprised.
The independence roots of $P_{3}^{11}$
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A fractals arises! And not just for this example - for every graph! We work with the *reduced independence polynomial* $f_G$ of a graph $G$ by deleting the constant term from the independence polynomial.
Definition 2 (BCN)  The independence fractal of a graph $G$ is the set

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Theorem 6 (BCN) The independence fractal $\mathcal{F}(G)$ of a graph $G \neq K_1$ is precisely the Julia set of its reduced independence polynomial (the boundary of the set of points with bounded forward orbits under $f$); this is equivalent to being the closure of the union of the roots of the reduced independence polynomial of powers of the graph.
The independence fractal $\mathcal{F}(K_1 \cup K_2)$
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One of the interesting questions you can ask about fractals is whether it is connected or not (and breaks up into fractal dust).
Theorem 7 (BCN) For any noncomplete graph $G$ and for all sufficiently large $p$, the join of $p$ copies of $G$ has a disconnected independence fractal.
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Theorem 8 (BCN) If $G$ is a graph with independence number 2 with at least 5 vertices and $\binom{n}{2} - m$ edges, then $\mathcal{F}(G)$ is a dusty subset of the interval $[-n/m,0]$. 
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Theorem 9 (BCN) The independence fractal of $aK_b$ is connected if $b = 2$ and $a$ is even, and totally disconnected otherwise.
The independence fractal $\mathcal{F}(3K_2)$
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These fractals have been extended to more abstract discrete structures, simplicial complexes [BHN].
One final remark about roots of independence polynomials. One can look at the average independence polynomial [BN],

\[ a_{ipn}(x) = \frac{1}{2^{\binom{n}{2}}} \sum_{G \in \mathcal{G}_n} i(G, x), \]

averaged over all graphs of order \( n \).
One final remark about roots of independence polynomials. One can look at the average independence polynomial [BN],

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averaged over all graphs of order \( n \).

It can be shown [BN] that while the independence polynomial of almost every graph of order \( n \) has a nonreal root, the average independence polynomials always have simple, real roots.
Now on to finite topologies:
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**Definition 3** Let $X$ be a set. A topology on $X$ is a collection $\mathcal{T}$ of subsets of $X$ that contain $\emptyset$, $X$, and is closed under finite intersections and arbitrary unions. A finite topology is one where $X$ is finite (and hence closed under finite intersections and unions).
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An example: Let $X = \{1, 2, 3, 4\}$, and take

$$\mathcal{T} = \{\emptyset, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}$$

Then it is straightforward to check that $\mathcal{T}$ defines a finite topology on $X$. 
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Now there is a bijection between finite topologies on a finite set $X$ and preorders (reflexive, transitive relations) on $X$, with open sets correspond to *ideals* (that is, downwards closed sets) in the preorder.
Open sets are the most fundamental and important concept in topology. For a finite topology, it is natural to count the number of open sets of each cardinality. Here we have another perfect opportunity to use generating functions!
Definition 4 Let $\mathcal{T}$ be a finite topology on set $X$, and $O_i$ be the number of open sets of size $i$ in $\mathcal{T}$. Then
\[
open(\mathcal{T}, x) = \sum O_i x^i
\]
is the open set polynomial of the finite topology.

Definition 5 Given a preorder $P$, the ideal polynomial of $P$ as
\[
ideal_P(x) = \sum_{k \geq 0} d_k x^k,
\]
where $d_k$ is the number of ideals of size $k$ in $P$.

From earlier, open set polynomials and ideal polynomials are one and the same (for corresponding finite topologies and preorders).
An example: Let \( X = \{1, 2, 3, 4\} \), and take the topology
\[
\mathcal{T} = \{\emptyset, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}
\]
on \( X \).

\[
\text{open}(\mathcal{T}, x) = 1 + x^2 + 2x^3 + x^4.
\]
Note that terms may be missing in the open set polynomial as a finite topology may not have open sets of each intermediate cardinality.
Question 3 What can we say about the sequence of numbers of open sets of each cardinality in a finite topology (ideals in a preorder)? What can be said about the roots of the open set polynomial (ideal polynomial)?
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Theorem 10 (BHTW) Let $P$ be a preorder on a set $X$ of cardinality $n$. If the width of $P$ is $w$, then the roots of $\text{ideal}_P(x) = \sum d_i x^i$ have modulus at most $\alpha = \frac{w\sqrt{2}}{\sqrt{2} - 1}$.
Question 3 What can we say about the sequence of numbers of open sets of each cardinality in a finite topology (ideals in a preorder)? What can be said about the roots of the open set polynomial (ideal polynomial)?

Theorem 11 (BHTW) Let $P$ be a preorder on a set $X$ of cardinality $n$. If the width of $P$ is $w$, then the roots of $\text{ideal}_P(x) = \sum d_i x^i$ have modulus at most $\alpha = \frac{w \sqrt{2}}{w \sqrt{2} - 1}$.

Corollary 1 Let $\sigma$ be a finite topology on a set $X$ of cardinality $n$. The roots of $\text{open}(\sigma, z)$ have modulus at most $2n$. 

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For width 1, we can find a tight upper bound on the moduli of the roots of ideal\(_P(x)\).

**Theorem 12** Let \(P\) be a preorder on a set of size \(n\) with width 1. Then all the preorder ideal roots of \(P\) have moduli at most \(\tau = (1 + \sqrt{5})/2\), and the result is best possible.
How large can the modulus of a root of a open set polynomial on a set $X$ of size $n$ be?

**Theorem 13 (BHTW)** For all $n \geq 2$ there is a finite $T_0$ topology with $n$ elements whose open set polynomial has a root of modulus at least

$$\frac{n}{\ln n - \ln \ln n} (1 + o(1)).$$
Finally, what about the distribution of the roots of open set polynomials in the complex plane?
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**Theorem 14 (BHTW)** *The closure of the open set polynomial roots is the entire complex plane.*
Open Problems:

**Question 4** When are all of the roots of an independence polynomial real? (By Seymour and Chudnovsky, it is when $G$ is claw–free.)

**Question 5** When are all of the roots of an open set polynomial real?

**Question 6** What is the true maximum modulus of an open set polynomial of a finite topology on a set $X$ of size $n$?

**Question 7** What is the independence fractal of a graph connected?
Question 8 What is the closure of the roots of open set polynomials of a finite $T_0$ topology on a set $X$ of size $n$?