Zeros of graph-counting polynomials and their accumulation sets

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1 Introduction

1.1 Potts model

• The $Q$-state Potts model is a generalization of the Ising model ($Q = 2$) with spin variable $\sigma = 1, \ldots, Q$ on each vertex.

• A connected graph $G = (V, E)$ is defined by its vertex (node) set $V$ and edge (bond) set $E$. Denote the number of vertices and edges as $n = |V|$ and $|E|$.

• Zero-field Hamiltonian $\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$.

• Potts Model Partition function:

$$Z(G, Q, v) = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}} = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} (1 + (a - 1) \delta_{\sigma_i \sigma_j}) = \sum_{G' \subseteq G} Q^{k(G')} v^{e(G')} ,$$

where $a = e^{\beta J} = v + 1$. $G' = (V, E')$ is a spanning subgraph of $G = (V, E)$ with $E' \subseteq E$. $k(G')$ and $e(G')$ denote the number of connected components and the number of edges of $G'$.

• Physical range of $v$: $v \geq 0$ for the ferromagnet; $-1 \leq v \leq 0$ for the antiferromagnet corresponding to $0 \leq T \leq \infty$.

$\bullet$ Physical range of $v$: $v \geq 0$ for the ferromagnet; $-1 \leq v \leq 0$ for the antiferromagnet corresponding to $0 \leq T \leq \infty$. 

$$0 \quad \text{AFM} \quad \infty \quad \text{FM} \quad 0 \quad T$$

-1 0 $\infty$ $v$
The ferromagnetic phase transition of the two-dimensional Potts model is second order for $0 \leq Q \leq 4$ and first order for $Q > 4$.

The (reduced) free energy per site in the limit $n = |V| \to \infty$: $$f(G, Q, v) = \lim_{n \to \infty} \ln(Z(G, Q, v)^{1/n}) .$$

Tutte polynomial, $T(G, x, y)$:
$$T(G, x, y) = \sum_{G' \subseteq G} (x - 1)^{k(G') - k(G)}(y - 1)^{c(G')} ,$$
where $c(G') = e(G') + k(G') - n(G')$ denotes the number of independent circuits.

Symmetry relation for a planar graph $G$:
$$T(G, x, y) = T(G^*, y, x) ,$$
where $G^*$ is the planar dual graph of $G$.

Relation between $Z(G, Q, v)$ and $T(G, x, y)$:
$$Z(G, Q, v) = (x - 1)^{k(G)}(y - 1)^{n(G)}T(G, x, y) ,$$
where $x = 1 + Q/v$, $y = 1 + v = a$. 
• The number of spanning trees of $G$: $N_{ST}(G) = T(G, 1, 1)$.

• The number of spanning forests of $G$: $N_{SF}(G) = T(G, 2, 1)$.

• The number of connected spanning subgraphs of $G$: $N_{CSSG}(G) = T(G, 1, 2)$.

• The number of spanning subgraphs of $G$: $N_{SSG}(G) = T(G, 2, 2) = 2^{e(G)}$.

• The zero temperature limit of the Potts antiferromagnet ($v = -1$) is equivalent to the chromatic polynomial:
  $$Z(G, Q, v = -1) = P(G, Q),$$
  which expresses the number of ways to color the vertices of the graph $G$ with $Q$ colors such that no two adjacent vertices have the same color.

• The ground state degeneracy per vertex of the $Q$-state Potts antiferromagnet is
  $$W(\{G\}, Q) = \lim_{|V|\to\infty} P(G, Q)^{1/|V|}.$$ 

• Relation between $T(G, x, y)$ and $P(G, Q)$:
  $$P(G, Q) = (-Q)^{k(G)}(-1)^{n(G)}T(G, x = 1 - Q, y = 0).$$
1.2 Flow polynomial

- Consider a connected graph $G = (V, E)$ with vertex set $V$ and edge set $E$.
- Assign an orientation to each of the edges in $G$.
- Consider an Abelian group $H$ of order $o(H) = Q$, represented as an additive group of integers mod $Q$.
- A flow on $G$ is defined as a mapping $E(G) \to H \setminus \{0\}$ that assigns to each of the oriented edges a nonzero element with Kirchhoff’s law satisfied at each vertex (the flow into each vertex is equal to the flow outward from this vertex).
- Nowhere-zero $Q$-flow: a flow must avoid zero flow numbers on any edge (since the assignment of the zero element to an edge is equivalent to the absence of the edge).
- The number of nowhere-zero $Q$-flow on a (connected) graph $G$ is given by the flow polynomial $F(G, Q)$, which is a special case of the Tutte polynomial or $Q$-state Potts model:

$$F(G, Q) = (-1)^{|E|-|V|+1} T(G, x = 0, y = 1 - Q)$$

$$= (-1)^{|E|} Q^{-n} Z(G, Q, -Q) .$$

- The degree of $F(G, Q)$ as a polynomial in $Q$ is $c(G)$, the cyclomatic number of $G$. 
• For a planar graph $G$,

$$F(G, Q) = Q^{-1} P(G^*, Q),$$

where $G^*$ is the planar dual graph of $G$.

• Define the function

$$fl(\{G\}, Q) = \lim_{|V| \to \infty} F(G, Q)^{1/f(G)},$$

where $f(G)$ is the number of faces of $G$. For a planar graph $G$, $f(G) = c(G) + 1$.

• It has the property

$$\lim_{Q \to \infty} \frac{fl(\{G\}, Q)}{Q} = 1,$$

which is the analogue to

$$\lim_{Q \to \infty} \frac{W(\{G\}, Q)}{Q} = 1.$$

• Depending on the family of strip graphs, the locus $B$ may or may not cross the real $Q$ axis. If it does, we denote the maximal point where it crosses this axis as $Q_{cf}(\{G\})$. 
1.3 Reliability polynomial

- A network is represented as a connected graph $G = (V, E)$.
- Each link independently has the same probability $p \in [0, 1]$ of operation (presence) and probability $1 - p$ of malfunction (absence).
- The all-terminal reliability polynomial $R(G, p)$ is defined as the probability that there is an operating communications link between any two nodes in the network, i.e. any two vertices in the set $V$ are connected.
- The contributions to $R(G, p)$ arise from the sum of connected spanning subgraphs $G' = (V, E')$ of $G$ with $E' \subseteq E$:
  
  $$R(G, p) = \sum_{G' \subseteq G} p^{|E'|}(1 - p)^{|E| - |E'|},$$

  which is a special case of the Tutte polynomial or $Q$-state Potts model:

  $$R(G, p) = p^{|V| - 1}(1 - p)^{|E| - |V| + 1}T(G, 1, \frac{1}{1 - p})$$

  $$= (1 - p)^{|E|} \lim_{Q \to 0} Q^{-1}Z(G, Q, v = \frac{p}{1 - p}).$$

- $Z(G, Q, v)$ always has an overall factor of $Q$, which cancels the factor of $Q^{-1}$. 
• $v = p/(1 - p)$ can also be expressed as

$$
p = 1 - e^{-K} = \frac{v}{1 + v}.
$$

The physical range of $p \in [0, 1]$ for the network corresponds to the physical range of temperature for the $Q = 0$ Potts ferromagnet:

$$
T = 0 \iff p = 1,
$$

$$
T = \infty \iff p = 0.
$$

The physical range of temperature for the Potts antiferromagnet would correspond to the unphysical interval $-\infty \leq p \leq 0$ for the network.

• General properties:

$$
R(G, 0) = 0, \quad R(G, 1) = 1,
$$

$$
R(G, p) \in [0, 1] \quad \text{and} \quad \frac{dR(G, p)}{dp} \geq 0 \quad \text{for} \quad p \in [0, 1].
$$

• Define the function

$$
r(G, p) = \lim_{|V| \to \infty} R(G, p)^{1/|V|}.
$$
1.4 Lattice types

- Denote the number of vertices in the transverse direction as $L_y$ and the number of vertices in the longitudinal direction as $L_x$ for strips.

(i) square lattice strip

(ii) triangular lattice strip

(iii) honeycomb lattice strip

(iv) self-dual strips of the square lattice
1.5 Boundary conditions

(i) \((F_{BC_y}, F_{BC_x}) = \text{free}\)

(ii) \((P_{BC_y}, F_{BC_x}) = \text{cylindrical}\)

(iii) \((F_{BC_y}, P_{BC_x}) = \text{cyclic}\)

(iv) \((F_{BC_y}, T_{PBC_x}) = \text{M"obius}\)

(v) \((P_{BC_y}, P_{BC_x}) = \text{toroidal}\)

(vi) \((P_{BC_y}, T_{PBC_x}) = \text{Klein bottle}\)

where \(TP\) denotes twisted periodic:
2 Potts model partition function on lattice strips

2.1 Structures of the Potts model partition function on lattice strips

- For recursively defined strips comprised of $L_x$ repeated subgraphs and periodic boundary condition in the longitudinal direction:

$$Z(G, Q, v) = \sum_{j=1}^{N_{Z,G,\lambda}} c_{Z,G,j} \left[ \lambda_{Z,G,j}(Q, v) \right]^{L_x},$$

$$P(G, Q) = \sum_{j=1}^{N_{P,G,\lambda}} c_{P,G,j} \left[ \lambda_{P,G,j}(Q) \right]^{L_x}.$$ 

- The coefficients for cyclic strips of the square, triangular and honeycomb lattices are polynomials in $Q$, and only one type of polynomial of each degree in $Q$ occurs:

$$c^{(d)} = U_{2d} \left( \frac{\sqrt{Q}}{2} \right) = \sum_{j=0}^{d} (-1)^j \binom{2d - j}{j} Q^{d-j} \quad \text{for } d \geq 0,$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind.
The first few of these coefficients are
\[ c^{(0)} = 1 , \quad c^{(1)} = Q - 1 , \]
\[ c^{(2)} = Q^2 - 3Q + 1 , \quad c^{(3)} = Q^3 - 5Q^2 + 6Q - 1 , \]
\[ c^{(4)} = (Q - 1)(Q^3 - 6Q^2 + 9Q - 1) . \]

The coefficients for the self-dual square lattice strip with width \( L_y \) are polynomials in \( Q \), and only one type of polynomial of each degree in \( Q \) occurs:
\[ \kappa^{(d)} = \sqrt{Q} \ U_{2d-1}(\sqrt{q}/2) = c^{(d)} + c^{(d-1)} \text{ for } d \geq 1 . \]

The first few of these coefficients are
\[ \kappa^{(1)} = Q , \quad \kappa^{(2)} = Q(Q - 2) , \]
\[ \kappa^{(3)} = Q(Q - 1)(Q - 3) , \quad \kappa^{(4)} = Q(Q - 2)(Q^2 - 4Q + 2) , \]
\[ \kappa^{(5)} = Q(Q^2 - 3Q + 1)(Q^2 - 5Q + 5) . \]
2.2 Example of $Z(G, Q, v)$

- $P(G, Q)$ for the cyclic and Möbius ladder graph (Biggs, Damerell and Sands, 1972):
  
  \[ P(sq, cyc, 2, Q) = (Q^2 - 3Q + 3)^{L_x} + (Q - 1)[(3 - Q)^{L_x} + (1 - Q)^{L_x}] + Q^2 - 3Q + 1 , \]
  
  \[ P(sq, Mb, 2, Q) = (Q^2 - 3Q + 3)^{L_x} + (Q - 1)[(3 - Q)^{L_x} - (1 - Q)^{L_x}] - 1 . \]

- $Z(G, Q, v)$ for the cyclic triangular strip with $L_y = 2$:
  
  \[ Z(tri, cyc, 2, Q, v) = c^{(2)}(v^2)^{L_x} + c^{(1)} \sum_{j=2}^{4} [\lambda_{Z,tri,cyc,2,j}(Q, v)]^{L_x} + \sum_{j=5}^{6} [\lambda_{Z,tri,cyc,2,j}(Q, v)]^{L_x} , \]

where $\lambda_{Z,tri,cyc,2,j}(Q, v)$ for $j = 2, 3, 4$ are the solutions of the cubic equation:

\[ \xi^3 - v(v^3 + 8v + 4v^2 + 2Q)\xi^2 + v^2(2v^3Q + 6v^3 + 8Qv + Q^2 + 8v^2 + 2v^4 + 6Qv^2)\xi \]
\[ -v^4(v + 1)^2(v + Q)^2 = 0 , \]

and $\lambda_{Z,tri,cyc,2,j}(Q, v)$ for $j = 5, 6$ are the solutions of the quadratic equation:

\[ \xi^2 - (v^4 + 4v^3 + 7v^2 + 4Qv + Q^2)\xi + (v + 1)^2(v + Q)^2v^2 = 0 . \]

In the physical region, the free energy is given by

\[ f(\{tri, cyc, 2\}, Q, v) = \frac{1}{2} \ln[\lambda_{Z,tri,cyc,2,5}(Q, v)] . \]
Figure 1: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 2: Zeros and $\mathcal{B}$ for the $4 \times \infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 3: Zeros and $\mathcal{B}$ for the $5 \times \infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
2.3 Dimension of the space of coloring configurations

- The dimension of the space of coloring configurations is equal to the sum of the multiplicities of each distinct eigenvalue:

\[ C_{P,G} = \sum_{j=1}^{N_{P,G,\lambda}} c_{P,G,j}, \quad C_{Z,G} = \sum_{j=1}^{N_{Z,G,\lambda}} c_{Z,G,j}. \]

- For the square or triangular lattice strips:

\[ C_{P,sq/tri,cyc,L_y} = P(T_{L_y}, Q) = Q(Q - 1)^{L_y-1}, \]

\[ C_{P,sq/tri,tor,L_y} = P(T_{\frac{L_y+1}{2}}, Q) \quad \text{for odd } L_y, \]

\[ C_{P,sq/tri,Kb,L_y} = 0, \]

\[ C_{Z,sq/tri,cyc,L_y} = C_{Z,sq/tri,tor,L_y} = Q^{L_y}, \]

\[ C_{Z,sq,Mb,L_y} = C_{Z,sq,Kb,L_y} = \begin{cases} Q^{L_y/2} & \text{for even } L_y \\ Q^{(L_y+1)/2} & \text{for odd } L_y \end{cases}. \]
2.4 Number of $\lambda$’s in the Potts model partition function

- For cyclic strips of the square, triangular and honeycomb lattices $\Lambda = sq, tri, hc$, define $n_{Z,\Lambda}(L_y, d) = n_{T,\Lambda}(L_y, d)$ as the number of terms $\lambda_{Z,\Lambda,cyc,L_y,j}$ in $Z(\Lambda, Q, v)$ that have as their coefficients $c_{Z,\Lambda,cyc,L_y,j} = c^{(d)}$.

- The dimension of the full transfer (coloring) matrix is given by

$$C_{Z,sq/tri/hc,cyc,L_y} = \sum_{d=0}^{L_y} n_{Z,sq/tri/hc}(L_y, d)c^{(d)} = Q^{L_y}.$$

- $n_{Z,sq/tri/hc}(L_y, d)$ with $d = 0, 1, ..L_y$ are determined as follows:

$$n_{Z,sq/tri/hc}(L_y, d) = 0 \quad \text{for} \quad d > L_y,$$

$$n_{Z,sq/tri/hc}(L_y, L_y) = 1, \quad n_{Z,sq/tri/hc}(1, 0) = 1,$$

with all other numbers $n_{Z,sq/tri/hc}(L_y, d)$ being determined by the two recursion relations:

$$n_{Z,sq/tri/hc}(L_y + 1, 0) = n_{Z,sq/tri/hc}(L_y, 0) + n_{Z,sq/tri/hc}(L_y, 1)$$

and

$$n_{Z,sq/tri/hc}(L_y + 1, d) = n_{Z,sq/tri/hc}(L_y, d - 1) + 2n_{Z,sq/tri/hc}(L_y, d)$$

$$+ n_{Z,sq/tri/hc}(L_y, d + 1) \quad \text{for} \quad L_y \geq 1 \text{ and } 1 \leq d \leq L_y + 1.$$
The number \( n_{Z, sq/tri/hc}(L_y, d) \) and their sum \( N_{Z, sq/tri/hc,cyc} \) (Saleur, 1990):

\[
n_{Z, sq/tri/hc}(L_y, d) = \frac{(2d + 1)}{(L_y + d + 1)} \left( \frac{2L_y}{L_y - d} \right), \quad N_{Z, sq/tri/hc,cyc} = \left( \frac{2L_y}{L_y} \right) \sim \pi^{-1/2} L_y^{-1/2} 4^{L_y}.
\]

Table 1: Table of numbers \( n_{Z, sq/tri/hc}(L_y, d) \) and their sums, \( N_{Z, sq/tri/hc,cyc} \), for cyclic strips of the square, triangular and honeycomb lattices. Blank entries are zero.

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• For cyclic strips, define $n_{P,\Lambda}(L_y, d)$ as the number of terms $\lambda_{P,\Lambda,cyc,L_y,j}$ in $P(\Lambda, Q)$ that have as their coefficients $c_{P,\Lambda,cyc,L_y,j} = c^{(d)}$.

• The dimension of the transfer (coloring) matrix for $\Lambda = sq/tri$ is given by

$$C_{P,sq/tri,cyc,L_y} = \sum_{d=0}^{L_y} n_{P,sq/tri}(L_y, d) c^{(d)} = Q(Q - 1)^{-1}.$$ 

• $n_{P,sq/tri}(L_y, d)$ with $d = 0, 1, \ldots L_y$ are determined as follows:

$$n_{P,sq/tri}(L_y, d) = 0 \quad \text{for} \quad d > L_y,$$

$$n_{P,sq/tri}(L_y, L_y) = 1, \quad n_{P,sq/tri}(1, 0) = 1,$$

with all other numbers $n_{P,sq/tri}(L_y, d)$ being determined by the two recursion relations:

$$n_{P,sq/tri}(L_y + 1, 0) = n_{P,sq/tri}(L_y, 1)$$

and

$$n_{P,sq/tri}(L_y + 1, d) = n_{P,sq/tri}(L_y, d - 1) + n_{P,sq/tri}(L_y, d) + n_{P,sq/tri}(L_y, d + 1)$$

for $L_y \geq 1$ and $1 \leq d \leq L_y + 1$. 
• The sum of the number $n_{P,sq/tri}(L_y, d)$:

$$N_{P,sq/tri,cyc} = 2(L_y - 1)! \sum_{j=0}^{[L_y/2]} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!} \sim L_y^{-1/2} 3^{L_y}.$$ 

Table 2: Table of numbers $n_{P,sq/tri}(L_y, d)$ and their sums, $N_{P,sq/tri,cyc}$, for cyclic strips of the square and triangular lattices. Blank entries are zero.

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• The dimension of the transfer (coloring) matrix for $\Lambda = hc$ is given by

$$C_{P, hc, cyc, L_y} = \sum_{d=0}^{L_y} n_{P, hc}(L_y, d)c^{(d)} = \begin{cases} Q(Q(Q - 1))^{L_y-1/2} & \text{for } L_y \text{ odd} \\ (Q(Q - 1))^{L_y/2} & \text{for } L_y \text{ even} \end{cases}.$$  

• $n_{P, hc}(L_y, d)$ with $d = 0, 1, \ldots L_y$ are determined as follows:

$$n_{P, hc}(L_y, d) = 0 \quad \text{for } d > L_y , \quad n_{P, hc}(L_y, L_y) = 1 ,$$

$$n_{P, hc}(2, 0) = 1 , \quad n_{P, hc}(2, 1) = 2 ,$$

with all other numbers $n_{P, hc}(L_y, d)$ being determined by the two recursion relations:

For even $L_y \geq 4$:

$$n_{P, hc}(L_y, d) = n_{P, hc}(L_y - 1, d - 1) + n_{P, hc}(L_y - 1, d) + n_{P, hc}(L_y - 1, d + 1) \quad \text{for } d \geq 1$$

while for $d = 0$, $n_{P, hc}(L_y, 0) = n_{P, hc}(L_y - 1, 1)$.

For odd $L_y \geq 3$:

$$n_{P, hc}(L_y, d) = n_{P, hc}(L_y - 1, d - 1) + 2n_{P, hc}(L_y - 1, d) + n_{P, hc}(L_y - 1, d + 1) \quad \text{for } d \geq 1$$

while for $d = 0$, $n_{P, hc}(L_y, 0) = n_{P, hc}(L_y - 1, 0) + n_{P, hc}(L_y - 1, 1)$. 
Table 3: Table of numbers $n_{P,hc}(L_y, d)$ and their sums, $N_{P,hc,cyc}$ for cyclic strips of the honeycomb lattice. Blank entries are zero.

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• For cyclic self-dual strips of the square lattice, define \( n_{Z,G_D}(L_y, d) \) as the number of terms \( \lambda_{Z,G_D,j} \) in \( Z(G_D, Q, v) \) that have as their coefficients \( c_{Z,G_D,j} = \kappa^{(d)} \).

• The dimension of the full transfer (coloring) matrix is given by

\[
C_{Z,G_D} = \sum_{d=1}^{L_y + 1} n_{Z,G_D}(L_y, d) \kappa^{(d)} = Q^{L_y + 1}.
\]

• The number \( n_{Z,G_D}(L_y, d) \) are

\[
n_{Z,G_D}(L_y, d) = \frac{2d}{L_y + d + 1} \left( \frac{2L_y + 1}{L_y - d + 1} \right).
\]

• Construct diagram with the entries in the even rows are given by \( n_{Z,sq/tri/hc}(L_y, d) \) while the entries in the odd rows are given by \( n_{Z,G_D}(L_y, d) \).

• The sum of the number \( n_{Z,G_D}(L_y, d) \), modulo the multiplicity \( \kappa^{(d)} \):

\[
N_{Z,G_D} = \sum_{d=1}^{L_y + 1} n_{Z,G_D}(L_y, d) = \binom{2L_y + 1}{L_y + 1} = \frac{1}{2} N_{Z,sq,cyc,L_y+1}
\]

\[
\sim \pi^{-1/2} L_y^{-1/2} 4^{L_y}.
\]
Table 4: Table of numbers $n_{Z,G_D}(L_y, d)$ and their sums, $N_{Z,G_D}$ for self-dual strips. Blank entries are zero.

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• For cyclic self-dual strips of the square lattice, define $n_{P,G_D}(L_y, d)$ as the number of terms $\lambda_{P,G_D,j}$ in $P(G_D, q)$ that have as their coefficients $c_{P,G_D,j} = \kappa^{(d)}$. In terms of the number of random walks $m(n, k)$,

$$n_{P,G_D}(L_y, d) = m(L_y, d - 1) .$$

• The dimension of the transfer (coloring) matrix is given by

$$C_{P,G_D} = \sum_{d=1}^{L_y+1} n_{P,G_D}(L_y, d) \kappa^{(d)} = Q(Q - 1)^{L_y} .$$

• The sum of the number $n_{P,G_D}(L_y, d)$, modulo the multiplicity $\kappa^{(d)}$:

$$N_{P,G_D} = \sum_{d=1}^{L_y+1} n_{P,G_D}(L_y, d) = L_y! \sum_{j=0}^{\left\lfloor \frac{L_y+1}{2} \right\rfloor} \frac{(L_y + 1 - j)}{(j!)^2(L_y + 1 - 2j)!} = \frac{1}{2} N_{P,sq,cyc,L_y+1}$$

$$\sim L_y^{-1/2} 3^{L_y} .$$
Table 5: Table of numbers $n_{P,G_D}(L_y, d)$ and their sums, $N_{P,G_D}$ for self-dual strips. Blank entries are zero.

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</table>
• For free strips of the triangular lattice (Jacobsen, Salas and Sokal, 2002):

\[ N_{Z,tri,free}(L_y) = C_{L_y} = \frac{1}{L_y + 1} \left( \frac{2L_y}{L_y} \right), \quad N_{P,tri,free}(L_y) = M_{L_y-1}, \]

where \( C_n \) is the Catalan number and \( M_n \) is the Motzkin number.

• For free strips of the square lattice:

\[ N_{Z,sq,free}(L_y) = \begin{cases} \frac{1}{2} \left[ C_{L_y} + \left( \frac{L_y}{L_y/2} \right) \right] & \text{for even } L_y \\ \frac{1}{2} \left[ C_{L_y} + \frac{1}{2} \left( \frac{L_y+1}{L_y+1/2} \right) \right] & \text{for odd } L_y \end{cases} \]

\[ N_{P,sq,free}(L_y) = \frac{1}{2} M_{L_y-1} + \frac{1}{2} (L'_y - 1)! \sum_{j=0}^{[L'_y/2]} \frac{(L'_y - j)}{(j!)^2(L'_y - 2j)!}, \quad \text{where } L'_y = \left\lfloor \frac{L_y + 1}{2} \right\rfloor. \]

• For cylindrical strips of the triangular lattice, the dimension can be reduced because of the translation symmetry:

\[ N_{Z,tri,cyl}(L_y) = \frac{1}{L_y} \left( C_{L_y} + \sum_{d|L_y; 1 \leq d < L_y} \phi(L_y/d) \binom{2d}{d} \right), \]

where \( d|L_y \) means that \( d \) divides \( L_y \) and \( \phi(n) \) is the Euler function, equal to the number of positive integers not exceeding the positive integer \( n \) and relatively prime to \( n \).
• For cylindrical strips of the square lattice, the dimension can be reduced because of both reflection and translation symmetries:

\[ N_{Z, sq, cyl}(L_y) = \frac{1}{2} \left( N_{tri, cyl}(L_y) + \left( \frac{L_y}{[L_y/2]} \right) \right) . \]

• For cylindrical strips of the triangular lattice, the dimension can be reduced from the number of non-crossing non-nearest-neighbor partitions of \( n \) vertices with periodic boundary conditions \( d_{L_y} \) because of the translation symmetry. We conjecture

\[ N_{P, tri, cyl}(L_y) = \frac{1}{L_y} \left( d_{L_y} + \sum_{d|L_y; \ 1 \leq d < L_y} \phi(L_y/d) \left( \frac{2d}{d} \right) \right) . \]

• For cylindrical strips of the square lattice, the dimension can be reduced from \( d_{L_y} \) because of both reflection and translation symmetries. We conjecture

\[ N_{P, sq, cyl}(L_y) = \begin{cases} \frac{1}{2} \left( N_{P, tri, cyl}(L_y) + \frac{1}{2} N_{P, sq/tri, cyc}(\frac{L_y}{2}) \right) & \text{for even } L_y \\ \frac{1}{2} \left( N_{P, tri, cyl}(L_y) + \frac{1}{4} N_{P, sq/tri, cyc}(\frac{L_y+1}{2}) - \frac{1}{2} R_{\frac{L_y-1}{2}} \right) & \text{for odd } L_y \geq 3 \end{cases} \]

where \( R_n \) is the Riordan number defined by \( R_0 = 1, R_1 = 0 \) and

\[ R_n = \sum_{j=1}^{n-1} (-1)^{n-j-1} M_j \quad \text{for } n \geq 2 . \]
Table 6: Dimension of the transfer (coloring) matrices for free and cylindrical strips of the square and triangular lattices.

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3 Flow polynomial

- The flow polynomial for a cyclic strip has the general form

\[
F(\Lambda[L_y \times m, cyc], x, y) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_{F,\Lambda}(L_y, d)} (\lambda_{F,\Lambda,L_y,d,j})^m,
\]

where the number of \( \lambda_{F,\Lambda,L_y,d,j} \) with coefficient \( c^{(d)} \) is denoted as \( n_{F,\Lambda}(L_y, d) \).

- For cyclic strips of the square and honeycomb lattice

\[
C_{F,sq/hc,L_y} = \sum_{d=0}^{L_y} n_{F,sq/hc}(L_y, d)c^{(d)} = (Q - 1)^{L_y}.
\]

- \( n_{F,sq/hc}(L_y, d) \), \( d = 0, 1, \ldots L_y \) are determined as follows:

\[
n_{F,sq/hc}(L_y, d) = 0 \quad \text{for} \quad d > L_y, \quad n_{F,sq/hc}(L_y, L_y) = 1, \quad n_{F,sq/hc}(1, 0) = 0,
\]

with all other numbers \( n_{F,sq/hc}(L_y, d) \) being determined by the two recursion relations

\[
n_{F,sq/hc}(L_y + 1, 0) = n_{F,sq/hc}(L_y, 1)
\]

and

\[
n_{F,sq/hc}(L_y + 1, d) = n_{F,sq/hc}(L_y, d - 1) + n_{F,sq/hc}(L_y, d) + n_{F,sq/hc}(L_y, d + 1)
\]

for \( L_y \geq 1 \) and \( 1 \leq d \leq L_y + 1 \).
• Summing the $n_{F, sq/hc}(L_y, d)$ over $d$ for a given strip with $L_y$, we obtain

$$N_{F, sq/hc,cyc} = \sum_{j=0}^{[L_y/2]} \left( \begin{array}{c} L_y \\ j \end{array} \right) \left( \begin{array}{c} L_y - j \\ j \end{array} \right) \sim L_y^{-1/2} 3^{L_y} ,$$

where $[x]$ is integer part of $x$.

• For the cases with noncompact accumulation set of the zeros $\mathcal{B}$, it is convenient to use the inverse variable $u = 1/Q$.

• For self-dual strips of the square lattice, the loci $\mathcal{B}$ are the same for the flow and chromatic polynomials.

• For (infinite) 2D lattices:

$$Q_{cf}(sq) = 3 , \quad Q_{cf}(tri) = \frac{3 + \sqrt{5}}{2} = 2.6180... ,$$

$$Q_{cf}(hc) = 4 , \quad Q_{cf}(diced) = 3 .$$
Table 7: Table of numbers $n_{F,sq/hc}(L_y, d)$ and their sums, $N_{F,sq/hc,cyc}$ for cyclic strips of the square and honeycomb lattices. Blank entries are zero.

<table>
<thead>
<tr>
<th>$L_y \setminus d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>$N_{F,sq/hc,cyc}$</th>
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<td></td>
<td></td>
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<td>9</td>
<td>1</td>
<td>8953</td>
</tr>
</tbody>
</table>
3.1 Square lattice strips

3.1.1 Free strips

- For $L_y = 2$ and $L_y = 3$, the zeros are discrete:
  
  $$F(sq[L_y = 2, L_x = m + 1, \text{free}], Q) = (Q - 1)(Q - 2)^{m-1},$$
  
  $$F(sq[L_y = 3, L_x = m + 1, \text{free}], Q) = (Q - 1)(Q - 2)(Q^2 - 5Q + 7)^{m-1}.$$  

- It is convenient to use a generating function to give the results for $L_y \geq 4$,

  $$\Gamma(G, Q, z) = \sum_{m=0}^{\infty} F(G_m, Q) z^m,$$

  where

  $$\Gamma(G, Q, z) = \frac{\mathcal{N}(G, Q, z)}{\mathcal{D}(G, Q, z)}$$

- For $L_y = 4$,

  $$\mathcal{N}(sq[L_y = 4, \text{free}], Q, z) = (Q - 1)(Q - 2)z[(Q - 2) - (Q - 1)(Q - 3)z],$$
  
  $$\mathcal{D}(sq[L_y = 4, \text{free}], Q, z) = 1 - (Q^3 - 8Q^2 + 24Q - 26)z$$
  
  $$-(Q^5 - 12Q^4 + 59Q^3 - 149Q^2 + 193Q - 101)z^2 + (Q - 2)^5(Q - 3)z^3.$$
Figure 4: Zeros and $\mathcal{B}$ for the $4 \times \infty$ strip of the square lattice with free boundary conditions in the $Q$ plane.
3.1.2 Cylindrical strips

- For $L_y = 2$ and $L_y = 3$, the zeros are discrete:

$$F(sq[L_y = 2, L_x = m + 1, cyl], Q) = (Q - 1)(Q - 2)^2(Q^2 - 3Q + 3)^{m-1}$$

$$= (Q - 1)(Q - 2)^2(Q^2 - 3Q + 3)^{m-1},$$

$$F(sq[L_y = 3, L_x = m+1, cyl], Q) = (Q - 1)(Q - 2)(Q - 3)^2(Q^3 - 6Q^2 + 14Q - 13)^{m-1}.$$

- For $L_y = 4$ and $L_y = 5$, the denominator of the generating function is the same as that for chromatic polynomials of the corresponding strips.
3.1.3 Cyclic and Möbius strips

- For $L_y = 2$ (dual to Read and Royle, 1991),
  \[
  F(sq[2 \times m, cyc], Q) = (Q - 2)^m + c^{(1)}(Q - 3)^m + c^{(2)}(-1)^m ,
  \]
  \[
  F(sq[2 \times m, Mb], Q) = (Q - 2)^m + c^{(1)}(Q - 3)^m - (-1)^m .
  \]

The locus $B$ divides the $Q$ plane into three regions. The closed region is bounded by
\[
Q = 3 + e^{i\theta} , \quad \text{for} \quad \frac{2\pi i}{3} \leq \theta \leq \frac{4\pi i}{3} \quad \text{and} \quad \frac{\pi i}{3} \leq \theta \leq \frac{\pi i}{3} .
\]

- For $L_y = 3$,
  \[
  F(sq[3 \times m, cyc], Q) = (\lambda_{sq,3,0,1})^m + c^{(1)} \sum_{j=1}^{3} (\lambda_{sq,3,1,j})^m + c^{(2)} \sum_{j=1}^{2} (\lambda_{sq,3,2,j})^m + c^{(3)} ,
  \]

where
\[
\lambda_{sq,3,0,1} = Q^2 - 5Q + 7 , \quad \lambda_{sq,3,1,1} = 2 - Q ,
\]
\[
\lambda_{sq,3,1,j} = \frac{1}{2}[10 - 6Q + Q^2 \pm (52 - 56Q + 28Q^2 - 8Q^3 + Q^4)^{1/2}] \quad \text{for} \quad j = 2, 3 ,
\]
\[
\lambda_{sq,3,2,1} = 2 - Q , \quad \lambda_{sq,3,2,2} = 4 - Q , \quad \lambda_{sq,3,3} = 1 .
\]
Figure 5: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 6: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 7: Zeros and $B$ for the $3 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $u$ plane.
Figure 8: Zeros and $\mathcal{B}$ for the $4 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
3.1.4 Toroidal and Klein bottle strips

- For $L_y = 2$,
  \[
  F(sq[2 \times L_x = m, tor], Q) = (Q^2 - 3Q + 3)^m + (Q - 1)(Q^2 - 4Q + 5)^m + (Q^2 - 3Q + 1)
  \]
  \[
  F(sq[2 \times L_x = m, Kb], Q) = (Q^2 - 3Q + 3)^m + (Q - 1)(Q^2 - 4Q + 5)^m - 1.
  \]

- For $L_y = 3$,
  \[
  F(sq[3 \times m, tor], Q) = \sum_{j=1}^{8} c_{sq,tor,3,j}(\lambda_{sq,tor,3,j})^m,
  \]
  where
  \[
  \lambda_{sq,tor,3,1} = Q^3 - 6Q^2 + 14Q - 13,
  \]
  \[
  \lambda_{sq,tor,3,(2,3)} = \frac{1}{2}[-18 + 16Q - 6Q^2 + Q^3 \pm \sqrt{R_{sq,tor,3}}] \quad \text{for } j = 2, 3,
  \]
  \[
  R_{sq,tor,3} = 256 - 440Q + 376Q^2 - 196Q^3 + 64Q^4 - 12Q^5 + Q^6,
  \]
  \[
  \lambda_{sq,tor,3,4} = Q-2, \quad \lambda_{sq,tor,3,5} = Q-1, \quad \lambda_{sq,tor,3,6} = Q-4, \quad \lambda_{sq,tor,3,7} = Q-5, \quad \lambda_{sq,tor,3,8} = -1.
  \]
  \[
  c_{sq,tor,3,1} = 1, \quad c_{sq,tor,3,2} = c_{sq,tor,3,3} = Q-1, \quad c_{sq,tor,3,4} = (Q + 1)(Q - 2),
  \]
  \[
  c_{sq,tor,3,5} = \frac{1}{2}c_{sq,tor,3,6} = \frac{1}{2}(Q-1)(Q-2), \quad c_{sq,tor,3,7} = \frac{1}{2}Q(Q-3), \quad c_{sq,tor,3,8} = Q^3-6Q^2+8Q-1.
  \]
Figure 9: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the square lattice with torus or Klein bottle boundary conditions in the $Q$ plane.
Figure 10: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the square lattice with torus or Klein bottle boundary conditions in the $u$ plane.
3.2 Triangular lattice strips

3.2.1 Free strips

- For $L_y = 2$,
  \[
  F(tri[L_x = 2, L_x = m, free], Q) = (Q - 1)(Q - 2)^{2(m-1)+1}.
  \]

- For $L_y = 3$,
  \[
  \lambda_{tri,free,L_y=3,j} = \frac{1}{2}[Q^4 - 7Q^3 + 21Q^2 - 33Q + 23 \pm \sqrt{R_{tri,free,3}}], \quad j = 1, 2,
  \]
  where
  \[
  R_{tri,free,3} = Q^8 - 14Q^7 + 91Q^6 - 360Q^5 + 949Q^4 - 1708Q^3 + 2047Q^2 - 1486Q + 497.
  \]
Figure 11: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the triangular lattice with free boundary conditions in the $Q$ plane.
3.2.2 Cyclic and Möbius strips

- For $L_y = 2$,
  \[
  F(tri[2 \times m, cyc], Q) = (Q - 2)^2m + c^{(1)}[(\lambda_{tri,2,1,1})^m + (\lambda_{tri,2,1,1})^m] + c^{(2)},
  \]
  where $\lambda_{tri,2,1,j} = \frac{1}{2}[6 - 4Q + Q^2 \pm (Q - 2)\sqrt{8 - 4Q + Q^2}]$ for $j = 1, 2$.

- For $L_y = 3$,
  \[
  F(tri[3 \times m, cyc], Q) = \sum_{j=1}^{2}(\lambda_{tri,3,0,j})^m + c^{(1)}\sum_{j=1}^{4}(\lambda_{tri,3,1,j})^m + c^{(2)}\sum_{j=1}^{3}(\lambda_{tri,3,2,j})^m + c^{(3)},
  \]
  where $\lambda_{tri,3,1,1} = Q^2 - 4Q + 5$, $\lambda_{tri,3,2,1} = Q^2 - 3Q + 3$,
  \[
  \lambda_{tri,3,0,j} = \frac{1}{2}[Q^4 - 7Q^3 + 21Q^2 - 33Q + 23 \pm (Q^8 - 14Q^7 + 91Q^6 - 360Q^5
  + 949Q^4 - 1708Q^3 + 2047Q^2 - 1486Q + 497)^{1/2}] \text{ for } j = 1, 2.
  \]
  The $\lambda_{tri,3,1,j}$ for $j = 2, 3, 4$ are the roots of the equation:
  \[
  \xi^3 - (Q^4 - 7Q^3 + 22Q^2 - 37Q + 30)\xi^2 + (Q^6 - 11Q^5 + 52Q^4 - 134Q^3
  + 200Q^2 - 168Q + 69)\xi - (2Q^4 - 15Q^3 + 42Q^2 - 51Q + 24) = 0,
  \]
  \[
  \lambda_{tri,3,2,j} = \frac{1}{2}[Q^2 - 5Q + 9 \pm (Q^4 - 10Q^3 + 43Q^2 - 90Q + 73)^{1/2}] \text{, } j = 2, 3.
Figure 12: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 13: Zeros and $B$ for the $2 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions in the $u$ plane.
Figure 14: Zeros and $B$ for the $3 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
3.2.3 Toroidal and Klein bottle strips

- For $L_y = 2$,

$$F(tri[2 \times m, tor], Q) = (\lambda_{tri,tor,2,1})^m + (\lambda_{tri,tor,2,2})^m + (Q - 1)[(\lambda_{tri,tor,2,3})^m + (\lambda_{tri,tor,2,4})^m]$$

$$+ \frac{1}{2} Q(Q - 3)2^m ,$$

where

$$\lambda_{tri,tor,2,j} = \frac{1}{2}[11 - 19Q + 15Q^2 - 6Q^3 + Q^4$$

$$\pm (129 - 446Q + 727Q^2 - 722Q^3 + 479Q^4 - 218Q^5 + 66Q^6 - 12Q^7 + Q^8)^{1/2} ] , \ j = 1, 2$$

$$\lambda_{tri,tor,2,j} = \frac{1}{2}[14 - 20Q + 15Q^2 - 6Q^3 + Q^4$$

$$\pm (212 - 600Q + 852Q^2 - 776Q^3 + 493Q^4 - 220Q^5 + 66Q^6 - 12Q^7 + Q^8)^{1/2} ] , \ j = 3, 4$$.
Figure 15: Zeros and $B$ for the $2 \times \infty$ strip of the triangular lattice with torus or Klein bottle boundary conditions in the $Q$ plane.
Figure 16: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the triangular lattice with torus or Klein bottle boundary conditions in the $u$ plane.
3.3 Honeycomb lattice strips

3.3.1 Cyclic and Möbius strips

- The $L_y = 2$ cyclic or Möbius strip of the honeycomb lattice can be constructed by inserting degree-2 vertices on each horizontal (longitudinal) edge of the square lattice. Because the insertion of degree-2 vertices does not affect the flow polynomial, it follows that, for $BC_x = FBC_x, PBC_x, TPBC_x$ and $L_y = 2$, we have

$$F(hc[2 \times L_x, FBC_y, BC_x], Q) = F(sq[2 \times L_x, FBC_y, BC_x], Q),$$

and

$$fl(hc[2 \times \infty, cyc/Mb], Q) = fl(sq[2 \times \infty, cyc/Mb], Q).$$

- For $L_y = 3$,

$$F(hc[3 \times m, cyc], Q) = (\lambda_{hc,3,0,1})^m + c^{(1)} \sum_{j=1}^{3} (\lambda_{hc,3,1,j})^m + c^{(2)} \sum_{j=1}^{2} (\lambda_{hc,3,2,j})^m + c^{(3)},$$

where

$$\lambda_{hc,3,0,1} = (Q - 3)^2, \quad \lambda_{hc,3,2,j} = \frac{1}{2} [7 - 2Q \pm \sqrt{13 - 4Q}] \quad \text{for} \quad j = 1, 2, \quad \lambda_{hc,3,3} = 1,$$

and the $\lambda_{hc,3,1,j}$, $j = 1, 2, 3$ are the roots of the equation

$$\xi^3 - (Q - 3)(Q - 5)\xi^2 - (Q - 3)^2(2Q - 5)\xi - (Q - 2)^2(Q - 3)^2 = 0.$$
Figure 17: Zeros and $B$ for the $3 \times \infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 18: Zeros and $B$ for the $3 \times \infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions in the $u$ plane.
4 Reliability polynomial

- For a recursive family of graphs $G_m$ comprised of $m$ repeated subgraph units, the reliability polynomial $R(G_m, p)$ has the general form

$$R(G_m, p) = p^{a_{1,G}m + a_{0,G}} \sum_{j=1}^{N_{R,G,\alpha}} c_{R,G,j}(\alpha_{G,j})^m,$$

where $a_{1,G} = L_y$ for the strips of the square and triangular lattices and $a_{1,G} = 2L_y$ for the strips of the honeycomb lattice.

- As the number of vertices $|V| \to \infty$, the asymptotic continuous accumulation set of the zeros of $R(G, p)$ in the complex $p$ plane is denoted $\mathcal{B}$.

- We define $p_c$ as the maximal point where $\mathcal{B}$ intersects the real axis. For the infinite-length limits of some lattice strips, the locus $\mathcal{B}$ does not cross the real axis, so no $p_c$ is defined. In certain of these cases, there are complex-conjugate arcs on $\mathcal{B}$ whose endpoints at $p_{end}, p^*_{end}$ are very close to the real axis; in these cases, it is useful to define a $(p_c)_{eff} = Re(p_{end})$.

- By the PM-FM phase transition point of the Potts model for the (infinite) 2D lattice,

$$p_c(sq) = \frac{4}{3}, \quad p_c(tri) = \frac{3}{2}.$$

- Singular loci $\mathcal{B}$ are independent of the longitudinal boundary conditions.
4.1 Square lattice strips

4.1.1 Free strips

- For $L_y = 1$ (tree graph $T_m$), $R(T_m, p) = p^{m-1}$ and $r = p$.
- For $L_y = 2$

$$\alpha_{sq,2,0,j} = \frac{1}{2} \left[ 4 - 3p \pm \sqrt{R_{sq,free,2}} \right], \quad j = 1, 2,$$

with

$$R_{sq,free,2} = 12 - 20p + 9p^2.$$  

The continuous accumulation set of zeros in the limit of infinite strip length, $m \to \infty$ is an arc of a circle with radius $1/3$ centered at $1$,

$$p = 1 + \frac{1}{3} e^{i\theta} \quad \theta \in [-\theta_{sq,free,2}, \theta_{sq,free,2}],$$

where

$$\theta_{sq,free,2} = \arctan(2\sqrt{2}) \simeq 70.53^\circ,$$

which crosses the real $p$ axis at $p_c = 4/3$.

- For $L_y = 3$, one can infer from the arc endpoints closest to the real axis that $(p_c)_{eff} \simeq 1.335$. 

Figure 19: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with free or cyclic or Möbius boundary conditions in the $p$ plane.
Figure 20: Zeros and $B$ for the $3 \times \infty$ strip of the square lattice with free or cyclic or Möbius boundary conditions in the $p$ plane.
4.1.2 Cyclic and Möbius strips

- For $L_y = 1$ (circuit graph $C_m$), $R(C_m, p) = p^{m-1}[m(1 - p) + p]$ and $r = p$.
  A zero of multiplicity $m - 1$ at $p = 0$ and another zero at $p = m/(m - 1)$.

- For $L_y = 2$,

\[
R(sq[L_y = 2, m, cyc], p) = p^{2m} \left[ (\alpha_{sq,2,0,1})^m + (\alpha_{sq,2,0,2})^m - 2(1 - p)^m \right.
\]
\[
+ mp^{-1}(1 - p)^2 \left\{ (\alpha_{sq,2,0,1})^{m-1} \left( 1 + \frac{3(1 - p)}{\sqrt{R_{sq,free,2}}} \right) \right.
\]
\[
+ (\alpha_{sq,2,0,2})^{m-1} \left( 1 - \frac{3(1 - p)}{\sqrt{R_{sq,free,2}}} \right) - (1 - p)^{m-1} \right\} ,
\]

\[
R(sq[L_y = 2, m, Mb], p) = p^{2m} \left[ (\alpha_{sq,2,0,1})^m + (\alpha_{sq,2,0,2})^m - (1 - p)^m \right.
\]
\[
+ mp^{-1}(1 - p)^2 \left\{ (\alpha_{sq,2,0,1})^{m-1} \left( 1 + \frac{3(1 - p)}{\sqrt{R_{sq,free,2}}} \right) \right.
\]
\[
+ (\alpha_{sq,2,0,2})^{m-1} \left( 1 - \frac{3(1 - p)}{\sqrt{R_{sq,free,2}}} \right) + (1 - p)^{m-1} \right\} .
\]
4.1.3 Cylindrical strips

- For $L_y = 2$,

$$\alpha_{sq,cyl,2,j} = \frac{1}{2} \left[ 6 - 8p + 3p^2 \pm \sqrt{(2 - p)(4 - 3p)(4 - 6p + 3p^2)} \right].$$

The locus $\mathcal{B}$ is given by the union of an arc of a circle with a line segment on the real axis:

$$\{p = 1 + \frac{1}{\sqrt{3}}e^{i\theta}, \ \theta \in [-\pi/2, \pi/2]\} \cup \{\frac{4}{3} \leq p \leq 2\}.$$

- For $L_y = 3$, $\alpha_{sq,cyl,3,j}$ with $1 \leq j \leq 3$ are roots of the equation

$$\xi^3 - (24 - 56p + 46p^2 - 13p^3)\xi^2 + (1 - p)^2(24 - 46p + 30p^2 - 7p^3)\xi - (1 - p)^5 = 0.$$

- For $L_y = 3$, $\mathcal{B}$ crosses the real $p$ axis at $p_c \simeq 1.402$.

- For $L_y = 4$, the locus includes two complex-conjugates arcs, a self-conjugate arc crossing the real axis at $p \simeq 1.364$, and a real line segment extending from $p = 4/3$ to $p \simeq 1.384$. 
Figure 21: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with cylindrical or torus or Klein bottle boundary conditions in the $p$ plane.
Figure 22: Zeros and $\mathcal{B}$ for the $3 \times \infty$ strip of the square lattice with cylindrical or torus or Klein bottle boundary conditions in the $p$ plane.
Figure 23: Zeros and $\mathcal{B}$ for the $4 \times \infty$ strip of the square lattice with cylindrical or torus or Klein bottle boundary conditions in the $p$ plane.
4.1.4 Toroidal and Klein bottle strips

- $L_y = 2$

\[
R(sq[L_y = 2, m, tor], p) = p^{2m} \left[ (\alpha_{sq,cyl,2,1})^m + (\alpha_{sq,cyl,2,2})^m - 2((1 - p)^2)^m \right] \\
- mp^{-1}(1 - p)^2 \left\{ \frac{(\alpha_{sq,cyl,2,1})^{m-1}}{2} \left( 2p - 3 + \frac{(p - 2)(6p^2 - 13p + 8)}{\sqrt{(p - 2)(3p - 4)(3p^2 - 6p + 4)}} \right) \right. \\
+ \frac{(\alpha_{sq,cyl,2,2})^{m-1}}{2} \left( 2p - 3 - \frac{(p - 2)(6p^2 - 13p + 8)}{\sqrt{(p - 2)(3p - 4)(3p^2 - 6p + 4)}} \right) + (1 - p)((1 - p)^2)^{m-1} \right\},
\]

\[
R(sq[L_y = 2, m, Kb], p) = p^{2m} \left[ (\alpha_{sq,cyl,2,1})^m + (\alpha_{sq,cyl,2,2})^m - ((1 - p)^2)^m \right] \\
- mp^{-1}(1 - p)^2 \left\{ \frac{(\alpha_{sq,cyl,2,1})^{m-1}}{2} \left( 2p - 3 + \frac{(p - 2)(6p^2 - 13p + 8)}{\sqrt{(p - 2)(3p - 4)(3p^2 - 6p + 4)}} \right) \right. \\
+ \frac{(\alpha_{sq,cyl,2,2})^{m-1}}{2} \left( 2p - 3 - \frac{(p - 2)(6p^2 - 13p + 8)}{\sqrt{(p - 2)(3p - 4)(3p^2 - 6p + 4)}} \right) - (1 - p)((1 - p)^2)^{m-1} \right\}
\]
4.2 Triangular lattice strips

4.2.1 Free strips

- For $L_y = 2$,

$$\alpha_{t,2,0,j} = \frac{1}{2} \left[ 7 - 10p + 4p^2 \pm (3 - 2p)\sqrt{5 - 8p + 4p^2} \right] \quad \text{for} \quad j = 1, 2.$$ 

The locus $\mathcal{B}$ is given by an arc of a circle:

$$\mathcal{B} : \quad p = 1 + \frac{1}{2}e^{i\theta}, \quad \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],$$

which crosses the real axis at $p_c = 3/2$.

4.2.2 Cyclic strips

- For $L_y = 2$,

$$R(t[L_y = 2, m, cyc], p) = p^{2m} \left[ (\alpha_{t,2,0,1})^m + (\alpha_{t,2,0,2})^m - 2((1 - p)^2)^m \right.$$

$$+ mp^{-1}(5 - 3p)^{-1}(1 - p)^2 \left\{ (\alpha_{t,2,0,1})^{m-1}(1 - p) \left( 7 - 4p + \frac{(3 - 2p)(5 - 4p)}{\sqrt{4p^2 - 8p + 5}} \right) \right.$$ 

$$+ (\alpha_{t,2,0,2})^{m-1}(1 - p) \left( 7 - 4p - \frac{(3 - 2p)(5 - 4p)}{\sqrt{4p^2 - 8p + 5}} \right) - 2(p^2 - 3p + 2)((1 - p)^2)^{m-1} \right]\right].$$
Figure 24: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the triangular lattice with free or cyclic or Möbius boundary conditions in the $p$ plane.
4.3 Honeycomb lattice strips

4.3.1 Free strips

- For $L_y = 2$,

$$\alpha_{hc,2,0,j} = \frac{1}{2} \left[ 6 - 5p \pm \sqrt{H_2} \right]$$

with

$$H_2 = 32 - 56p + 25p^2.$$ 

The locus $B$ is given by an arc of a circle:

$$B : \quad p = 1 + \frac{1}{5} e^{i\theta}, \quad \theta \in [-\theta_{hc,2}, \theta_{hc,2}] ,$$

where

$$\theta_{hc,2} = \arctan(4/3) \simeq 53.13^\circ ,$$

which crosses the real axis at $p_c = 6/5$. 
Figure 25: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the honeycomb lattice with free or cyclic or Möbius boundary conditions in the $p$ plane.
4.3.2 Cyclic and Möbius strips

- For $L_y = 2$,

$$R(hc[L_y = 2, m, cyc], p) = p^{4m}[(\alpha_{hc,2,0,1})^m + (\alpha_{hc,2,0,2})^m - 2(1-p)^m$$

$$+ mp^{-1}(1-p)^2 \left\{ (\alpha_{hc,2,0,1})^{m-1} \left( 3 + \frac{16 - 15p}{\sqrt{H_2}} \right) \right.$$ 

$$+ (\alpha_{hc,2,0,2})^{m-1} \left( 3 - \frac{16 - 15p}{\sqrt{H_2}} \right) - 2(1-p)^{m-1} \right\} ] ,$$

$$R(hc[L_y = 2, m, Mb], p) = p^{4m}[(\alpha_{hc,2,0,1})^m + (\alpha_{hc,2,0,2})^m - (1-p)^m$$

$$+ mp^{-1}(1-p)^2 \left\{ (\alpha_{hc,2,0,1})^{m-1} \left( 3 + \frac{16 - 15p}{\sqrt{H_2}} \right) \right.$$ 

$$+ (\alpha_{hc,2,0,2})^{m-1} \left( 3 - \frac{16 - 15p}{\sqrt{H_2}} \right) + 2(1-p)^{m-1} \right\} ] .$$
4.4 Self-dual strips of the square lattice

- The reliability polynomial of the self-dual strip graph is

\[ R(G_D[L_y \times m], p) = p^{L_y m} \sum_{d=1}^{L_y + 1} (-1)^{d+1} d \sum_{j=1}^{n_{T,G_D}(L_y,d)} (\alpha_{G_D,L_y,d,j})^m, \]

where

\[ \alpha_{G_D,L_y,d,j} = (1 - p)^{L_y} \lambda_{T,G_D,L_y,d,j} \bigg|_{x=1,y=1/(1-p)}. \]

- For \( L_y = 1 \) (wheel graph),

\[ R(G_D[1 \times m], p) = p^m [(\alpha_{G_D,1,1,1})^m + (\alpha_{G_D,1,1,2})^m - 2(1 - p)^m], \]

where

\[ \alpha_{G_D,1,1,j} = \frac{1}{2} [3 - 2p \pm \sqrt{5 - 8p + 4p^2}] \quad j = 1, 2. \]

The locus \( \mathcal{B} \) forms arc of a circle centered at \( p = 1 \) of radius 1/2 with endpoints at \( p = 1 \pm (1/2)i \), i.e.,

\[ p = 1 + \frac{1}{2} e^{i\theta}, \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \]

which crosses the real axis at \( p_c = 3/2 \). This locus is the same as that for the triangular lattice with free transverse boundary conditions.
For \( L_y = 2 \),

\[
R(G_D[2 \times m], p) = p^{2m}[\sum_{j=1}^{5} (\alpha_{G_D,2,1,j})^m - 2 \sum_{j=1}^{4} (\alpha_{G_D,2,2,j})^m + 3((1 - p)^2)^m],
\]

where \( \alpha_{sqdbc,2,2,j} \) are the roots of the degree-4 equation:

\[
\xi^4 + a_{G_D,2,1,1}\xi^3 + a_{G_D,2,2,1}\xi^2 + a_{G_D,2,3,1}\xi + a_{G_D,2,4,1} = 0
\]

with

\[
\begin{align*}
a_{G_D,2,1,1} &= -(1 - p)(7 - 5p), \\
a_{G_D,2,2,1} &= (1 - p)^2(13 - 17p + 5p^2), \\
a_{G_D,2,3,1} &= -(1 - p)^4(7 - 5p), \\
a_{G_D,2,4,1} &= (1 - p)^6.
\end{align*}
\]

\( \alpha_{G_D,2,1,j} \) are the roots of the degree-5 equation:

\[
\xi^5 + b_{G_D,2,1,1}\xi^4 + b_{G_D,2,2,1}\xi^3 + b_{G_D,2,3,1}\xi^2 + b_{G_D,2,4,1}\xi + b_{G_D,2,5,1} = 0
\]

with

\[
\begin{align*}
b_{G_D,2,1,1} &= -(12 - 18p + 7p^2), \\
b_{G_D,2,2,1} &= (1 - p)(36 - 67p + 41p^2 - 8p^3), \\
b_{G_D,2,3,1} &= -(1 - p)^2(36 - 75p + 53p^2 - 13p^3), \\
b_{G_D,2,4,1} &= 2(1 - p)^4(3 - 2p)(2 - p), \\
b_{G_D,2,5,1} &= -(1 - p)^6.
\end{align*}
\]

The locus \( B \) consists of arcs, again concave to the left, that almost cross, but actually have endpoints near to, the real axis at \( q \approx 1.4 \).

For \( L_y = 3 \), the locus crosses the real axis at \( q \approx 1.384 \).
Figure 26: Zeros and $B$ for the $1 \times \infty$ self-dual strip of the square lattice in the $p$ plane.
Figure 27: Zeros and $\mathcal{B}$ for the $2 \times \infty$ self-dual strip of the square lattice in the $p$ plane.
Figure 28: Zeros and $B$ for the $3 \times \infty$ self-dual strip of the square lattice in the $p$ plane.
4.5 Recursive families of lattice strip graphs with zeros and $\mathcal{B}$ outside the disk $|p - 1|

Graphs with $|p - 1| > 1$ (violating the Brown-Colbourn conjecture) were first found by Royle and Sokal, 2003. We construct recursive families of graphs whose $L_x \to \infty$ limits have infinitely many zeros with $|p - 1| > 1$.

4.5.1 $L_y = 2$ free strip of the $sq_d$ lattice with multiple transverse edges

- Consider strips of the $sq_d$ lattice, $sq_d(2 \times L_x, \text{free})$, which are strips of the square lattice with diagonal edges (equivalent to the Potts model on the square lattice with next-nearest-neighbor spin-spin couplings), and replace each vertical edge by six edges joining the same pair of vertices.

- The tips of two complex-conjugate arcs of $\mathcal{B}$ end at the points

$$p = 0.327752 \pm 0.747464i,$$

which have

$$|p - 1| = 1.005296.$$
Figure 29: Zeros and $\mathcal{B}$ for $L_y = 2$ free strip of the $sq_d$ lattice with multiple transverse edges in the $p$ plane.
4.5.2 $L_y = 2$ free strip of the $sq_d$ lattice with multiple longitudinal edges

- Consider strips of the $sq_d$ lattice, $sq_d(2 \times L_x, \text{free})$, and replace each horizontal edge by six edges joining the same pair of vertices.

- The endpoints of the arcs that extend outside the disk $|p - 1| \leq 1$ are
  \[ p = 0.4346475 \pm 0.8266808i , \]
  which have
  \[ |p - 1| = 1.001511 . \]
Figure 30: Zeros and $\mathcal{B}$ for $L_y = 2$ free strip of the $sq_d$ lattice with multiple longitudinal edges in the $p$ plane.
4.5.3 Free strip comprised of $K_4$ subgraphs intersecting on common edges with multiple edges

- Consider strip graph comprised of $K_4$ subgraphs intersecting on common edges ($K_2$’s) with 6-fold replicated edges.

- The tips of two complex-conjugate arcs extend outside the disk $|p - 1| \leq 1$, ending at the points

  $$p = 0.4254334 \pm 0.8216255i,$$

  which have

  $$|p - 1| = 1.002594.$$
Figure 31: Zeros and $\mathcal{B}$ for free strip comprised of $K_4$ subgraphs intersecting on common edges with multiple edges in the $p$ plane.
5 Partition function zeros of a restricted Potts model

- The phase transition temperatures of the ferromagnetic Potts model (Baxter):
  - Square lattice: \( Q = v^2 \),
  - Triangular lattice: \( Q = v^2(v + 3) \),
  - Honeycomb lattice: \( Q^2 + 3Qv - v^3 = 0 \).

- For the square lattice, \( Q = v^2 \) is a self-dual point such that
  \[
  Z(G, Q, v) \propto Z(G^*, Q, v_d) ,
  \]
  where \( G^* \) denotes the planar dual to \( G \) and \( v_d = \frac{Q}{v} \).

- The Tutte-Beraha numbers:
  \[
  Q_r = 4 \cos^2(\pi/r) \quad \text{for} \quad 1 \leq r \leq \infty .
  \]

- \( Q_1 = 4, Q_2 = 0, Q_3 = 1, Q_4 = 2, Q_5 = (3 + \sqrt{5})/2, Q_6 = 3 \).

- \( Q_r \) increases monotonically from 0 to 4 as \( r \) increases from 2 to \( \infty \).

- Partition function for lattice \( \Lambda \) strips with \( BC \) boundary conditions:
  \[
  Z(\Lambda[L_y \times L_x, BC], Q, v) = \sum_j c_j (\lambda_{\Lambda,BC,L_y,j})^{L_x}
  \]
  where the coefficients \( c_j \) are independent of \( L_x \), and \( \lambda_{\Lambda,BC,L_y,j} = v^{L_y} \tilde{\lambda}_{\Lambda,BC,L_y,j} \).
5.1 Square lattice strips

5.1.1 Free strips

• For $L_y = 1$, $Z(sq[1 \times m, \text{free}], v^2, v) = v^{m+1}(v + 1)^{m-1}$. Zeros are at the two discrete points $v = 0$ and $v = -1$.

• For $L_y = 2$,
  
  \[ \lambda_{sq,2,0,j} = \frac{1}{2}[(v + 2)^2 \pm \sqrt{v^4 + 4v^3 + 12v^2 + 20v + 12}] , \]

  where $j = 1, 2$ correspond to $\pm$.

5.1.2 Cylindrical strips

• For $L_y = 2$,
  
  \[ \lambda_{sq,cyl,2,j} = \frac{1}{2}[3v^2 + 8v + 6 \pm (v + 2)\sqrt{5v^2 + 12v + 8}] , \]

  where $j = 1, 2$ correspond to $\pm$. 
Figure 32: Zeros and $B$ for the $2 \times \infty$ strip of the square lattice with free boundary conditions in the $v$ plane.
Figure 33: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with free boundary conditions in the $Q$ plane.
Figure 34: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with cylindrical boundary conditions in the $\nu$ plane.
Figure 35: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with cylindrical boundary conditions in the $Q$ plane.
5.1.3 Cyclic and Möbius strips

• For $L_y = 1$
  
  $\mathcal{Z}(C_m, v^2, v) = v^m(v + 1)[(v + 1)^{m-1} + v - 1]$.

  $\mathcal{B}_v$ is the circle $|v + 1| = 1$, i.e., $v = -1 + e^{i\phi}$ for $0 \leq \phi \leq 2\pi$.

  $\mathcal{B}_Q$ is given by $Q = 2 \cos \phi (\cos \phi - 1) + 2 \sin \phi (\cos \phi - 1)i$ for $0 \leq \phi \leq 2\pi$.

• For $L_y = 2$, there are six $\lambda$: $\lambda_{sq, 2, 0, j}$ with $j = 1, 2$, $\lambda_{sq, 2, 2, 1} = 1$, and

  $\lambda_{sq, 2, 1, 1} = 1 + v$, \quad $\lambda_{sq, 2, 1, j} = v + 2 \pm \sqrt{2v + 3}$, where $j = 2, 3$ correspond to $\pm$.

5.1.4 Toroidal and Klein bottle strips

• For $L_y = 2$, there are six $\lambda$: $\lambda_{sq, tor, 2, j} = \lambda_{sq, cyl, 2, j}$ with $j = 1, 2$,

  $\lambda_{sq, tor, 2, j} = \frac{1}{2}\{(v + 2)(v + 3) \pm [(v^2 + 3v + 8 + 4\sqrt{2})(v^2 + 3v + 8 - 4\sqrt{2})]^{1/2}\}$, where $j = 3, 4$ correspond to $\pm$, and $\lambda_{sq, tor, 2, 5} = v + 1$, $\lambda_{sq, tor, 2, 6} = 1$. 
Figure 36: $B$ for the $1 \times \infty$ strip of the square lattice with cyclic boundary conditions in the $Q$ plane.
Figure 37: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $v$ plane.
Figure 38: Zeros and $B$ for the $2 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
Figure 39: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with toroidal or Klein bottle boundary conditions in the $v$ plane.
Figure 40: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the square lattice with toroidal or Klein bottle boundary conditions in the $Q$ plane.
5.2 Triangular lattice strips

- At $Q = Q_r$, the roots of $Q = v^2(v + 3)$ are
  
  $v_{t1}(r) = -1 + 2 \cos\left(\frac{2(r + 1)\pi}{3r}\right)$,  
  $v_{t2}(r) = -1 + 2 \cos\left(\frac{2(r - 1)\pi}{3r}\right)$,  
  $v_{t3}(r) = -1 + 2 \cos\left(\frac{2\pi}{3r}\right)$.

- For $r = 2$, $Q = Q_2 = 0$,
  
  $v_{t1}(2) = -3$,  
  $v_{t2}(2) = 0$,  
  $v_{t3}(2) = 0$.

- For $r = 4$, $Q = Q_4 = 2$,
  
  $v_{t1}(4) = -1 - \sqrt{3}$,  
  $v_{t2}(4) = -1$,  
  $v_{t3}(4) = -1 + \sqrt{3}$.

- For $r = 6$, $Q = Q_6 = 3$,
  
  $v_{t1}(6) \simeq -2.532089$,  
  $v_{t2}(6) \simeq -1.347296$,  
  $v_{t3}(6) \simeq 0.879385$.

- For $r = \infty$, $Q = Q_\infty = 4$,
  
  $v_{t1}(\infty) = v_{t2}(\infty) = -2$,  
  $v_{t3}(\infty) = 1$. 
5.2.1 Free strips

• For $L_y = 2$,

$$
\bar{\lambda}_{t,2,0,j} = \frac{(v + 1)}{2} \left[ v^3 + 5v^2 + 9v + 7 \pm (v + 3)\sqrt{(v + 1)(v^3 + 3v^2 + 3v + 5)} \right],
$$

where $j = 1, 2$ correspond to $\pm$.

5.2.2 Cyclic and Möbius strips

• For $L_y = 2$, there are six $\bar{\lambda}$: $\bar{\lambda}_{t,2,0,j}$ with $j = 1, 2$, $\bar{\lambda}_{t,2,2,1} = 1$, and $\bar{\lambda}_{t,2,1,j}$ with $j = 1, 2, 3$ are solutions to the equation

$$
\eta^3 - (v + 2)(3v + 4)\eta^2 + (v + 2)(v + 1)(3v^2 + 9v + 4)\eta - (v + 1)^2(v^2 + 3v + 1)^2 = 0.
$$

• For $L_y = 2$, the loci consists of closed curves that intersect the real $v$ and $Q$ axis at:

(i) $v = 0$ and $v = -3 \implies Q = Q_2 = 0$,
(ii) $v = -1 \implies Q = Q_4 = 2$,
(iii) $v = -2 \implies Q = 4$.

• From the calculations for $L_y = 3, 4, 5$, we find that $B_v$ crosses the negative real axis at $v_{t2}(2\ell)$, equivalently $B_Q$ crosses the real axis at $Q_{2\ell}$, for $1 \leq \ell \leq L_y$. 

Figure 41: Zeros and $B$ for the $2 \times \infty$ strip of the triangular lattice with free boundary conditions in the $v$ plane.
Figure 42: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the triangular lattice with free boundary conditions in the $Q$ plane.
Figure 43: Zeros and $\mathcal{B}$ for the $2 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions in the $v$ plane.
Figure 44: Zeros and $B$ for the $2 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions in the $Q$ plane.
5.3 Honeycomb lattice strips

- At $Q = Q_r$, the roots of $Q^2 + 3Qv - v^3 = 0$ are

  
  \[
  v_{hc1}(r) = -4 \cos\left(\frac{\pi}{r}\right) \cos\left[\frac{\pi}{3}\left(\frac{1}{r} - 1\right)\right],
  \]

  
  \[
  v_{hc2}(r) = -4 \cos\left(\frac{\pi}{r}\right) \cos\left[\frac{\pi}{3}\left(\frac{1}{r} + 1\right)\right],
  \]

  
  \[
  v_{hc3}(r) = 4 \cos\left(\frac{\pi}{r}\right) \cos\left(\frac{\pi}{3r}\right).
  \]

- For $L_y = 2$ cyclic strip, the points (i) $v = 0 \Rightarrow Q = Q_2 = 0$ and (ii) $v = -2 \Rightarrow Q = Q_4 = 2$, and $Q = 4$ are on the respective loci $\mathcal{B}_v$ and $\mathcal{B}_Q$.

- For $L_y = 3$ cyclic strip, in addition to the above points, $v_{hc1}(6) = -2\sqrt{3} \cos(5\pi/18) \Rightarrow Q = Q_6 = 3$, is on the respective loci $\mathcal{B}_v$ and $\mathcal{B}_Q$.

- For cyclic strips up to width $L_y = 5$, we find that $\mathcal{B}_v$ crosses the real axis at $v = v_{hc1}(2\ell)$, equivalently $\mathcal{B}_Q$ crosses the real axis at $Q_{2\ell}$, for $1 \leq \ell \leq L_y$. 
5.4 Self-dual strips of the square lattice

- The Potts model partition function has the general form

\[
Z(G_D[L_y \times L_x], Q, v) = Q v^{L_x L_y} \sum_{d=1}^{L_y+1} \sum_{j=1}^{n_{Z,G_D}(L_y,d)} \bar{\kappa}^{(d)}(\bar{\lambda}_{L_y,d,j})^{L_x},
\]

where

\[
\bar{\kappa}^{(d)} = \prod_{k=1}^{d-1} (Q - s_{d,k}), \quad \text{with } s_{d,k} = 4 \cos^2 \left( \frac{\pi k}{2d} \right).
\]

- For \( L_y = 1 \),

\[
Z(G_D[1 \times L_x], v^2, v) = v^{m+2}[(\bar{\lambda}_{1,1,1})^{L_x} + (\bar{\lambda}_{1,1,2})^{L_x} + \bar{\kappa}^{(2)}],
\]

where

\[
\bar{\lambda}_{1,1,j} = \frac{1}{2} [3 + 2v \pm \sqrt{5 + 4v}]
\]

with \( j = 1, 2 \) corresponding to \( \pm \).
Figure 45: Zeros and $\mathcal{B}$ for the $1 \times \infty$ self-dual strips of the square lattice in the $v$ plane.
Figure 46: Zeros and $B$ for the $1 \times \infty$ self-dual strips of the square lattice in the $Q$ plane.
• For $L_y = 2$,
  - All ten $\tilde{\lambda}$ have unit modulus at $v = -2$.
  - In the interval $-2 \leq v \leq -1$, the locus $B_v$ crosses the negative real $v$ axis at $v = -2$ and $v = -2 \cos(\pi/(2\ell + 1))$ for $\ell = 1, 2$,
    i.e., $-2, -1, \text{ and } -2 \cos(\pi/5) = -\left(1 + \sqrt{5}\right)/2 \simeq -1.618$.
  - Correspondingly, in the interval $1 \leq Q \leq 4$ the locus $B_Q$ crosses the positive $Q$ axis at $4 \cos^2(\pi/(2\ell + 1))$ for $\ell = 0, 1, 2$,
    i.e., $4, 1, \text{ and } 4 \cos^2(\pi/5) = \left(3 + \sqrt{5}\right)/2 \simeq 2.618$.
• For $L_y = 3$,
  - In the interval $-2 \leq v \leq -1$, the locus $B_v$ crosses the negative real $v$ axis at $v = -2$ and $v = -2 \cos(\pi/(2\ell + 1))$ for $\ell = 1, 2, 3$,
    i.e., $-2, -1, -2 \cos(\pi/5) \simeq -1.618, \text{ and } -2 \cos(\pi/7) \simeq -1.802$.
  - Correspondingly, in the interval $1 \leq Q \leq 4$ the locus $B_Q$ crosses the positive $Q$ axis at $4 \cos^2(\pi/(2\ell + 1))$ for $\ell = 0, 1, 2, 3$,
    i.e., $4, 1, 4 \cos^2(\pi/5) \simeq 2.618, \text{ and } 4 \cos^2(\pi/7) \simeq 3.247$. 
Figure 47: Zeros and \( B \) for the \( 2 \times \infty \) self-dual strips of the square lattice in the \( v \) plane.
Figure 48: Zeros and $\mathcal{B}$ for the $2 \times \infty$ self-dual strips of the square lattice in the $Q$ plane.
Figure 49: Zeros and $\mathcal{B}$ for the $2 \times \infty$ self-dual strips of the square lattice in the $1/Q$ plane.
Figure 50: Zeros and $\mathcal{B}$ for the $3 \times \infty$ self-dual strips of the square lattice in the $v$ plane.
Figure 51: Zeros and $\mathcal{B}$ for the $3 \times \infty$ self-dual strips of the square lattice in the $Q$ plane.
Figure 52: Zeros and $\mathcal{B}$ for the $3 \times \infty$ self-dual strips of the square lattice in the $1/Q$ plane.
6 Zeros of the Potts model partition function in the large-$Q$ limit

6.1 Approximation for the locus of partition zeros for large $Q$

- Arrange the terms of the partition function in order of power of $v$:

$$Z(G, Q, v) = \sum_{j=0}^{e(G)} c_j(Q)v^j,$$

where the coefficients $c_j(Q)$ are polynomials in $Q$ containing $\binom{e(G)}{j}$ terms.

- Denote $\delta(G)$ as the minimal degree of a vertex in $G$. The first few terms with high powers of $v$ are given by $\binom{e(G)}{j} Q v^{e(G)-j}$ for $j < \delta(G)$.

- Denote $g(G)$ as the girth of $G$. The first few terms with low powers of $v$ are given by $\binom{e(G)}{j} Q^{n(G)-j} v^j$ for $j < g(G)$.

- Arrange the terms as descending powers of $v$:

$$Z(G, Q, v) = Q v^{e(G)} + e(G)Q v^{e(G)-1} + \binom{e(G)}{2} Q v^{e(G)-2} + ...$$

$$+ \binom{e(G)}{2} Q^{n(G)-2} v^2 + e(G)Q^{n(G)-1} v + Q^{n(G)}.$$
• Denote $\kappa_\Lambda$ as the coordination number. In the thermodynamic limit,
\[
\kappa_\Lambda = 2 \lim_{n(G) \to \infty} \frac{e(G)}{n(G)} ,
\]
which is independent of boundary conditions.

• Define a new variable $x_G = \frac{v}{Q^{n(G)/e(G)}}$. In the thermodynamic limit, $x_\Lambda = \frac{v}{Q^{2/\kappa_\Lambda}}$.

• For the square, kagomé, triangular and honeycomb lattices:
\[
\kappa_{sq} = \kappa_{kag} = 4 , \quad \kappa_t = 6 , \quad \kappa_{hc} = 3 .
\]
\[
x_{sq} = x_{kag} = \frac{v}{\sqrt{Q}} , \quad x_t = \frac{v}{Q^{1/3}} , \quad x_{hc} = \frac{v}{Q^{2/3}} .
\]

• Truncation of the full partition function with $Q$ large:
\[
Z(G, Q, v)_\text{trunc.} = Qv^{e(G)} + Q^{n(G)} = Q^{n(G)+1} [x_G^{e(G)} + Q^{-1}] .
\]

• The zeros of this truncation are given by $x_G = (-Q^{-1})^{1/e(G)}$.

• Consider $e(G) \to \infty$ and $n(G) \to \infty$ with the ratio $e(G)/n(G) = \kappa_G/2$ finite. If $Q \to \infty$, but more slowly than $\exp(be(G))$ with $b$ a real positive constant,
\[
\lim_{Q \to \infty} \lim_{e(G) \to \infty} |Q|^{1/e(G)} = 1 ,
\]
and these zeros merge onto the unit circle $|x_G| = 1$.  

6.2 Partition function zeros for finite lattice sections

6.2.1 General structure

• Reexpress the truncation of the partition function as

\[ Z(G, Q, v)_{\text{trunc.}} = Q v^{e(G)} + Q^{n(G)} = Q^{(2e(G)/\kappa_\Lambda)+1} \left[ x_\Lambda^{e(G)} + Q^{n(G)-(2e(G)/\kappa_\Lambda)-1} \right]. \]

• The zeros of this approximation to \( Z(G, Q, v) \) are located on a circle in the \( x_\Lambda \) plane with radius

\[ r(\Lambda, BC) = Q^{p(\Lambda, BC)} \]

where

\[ p(\Lambda, BC) = \frac{n(G) - 1}{e(G)} - \frac{2}{\kappa_\Lambda}. \]

• The radius \( r(\Lambda, BC) \) is less (greater) than unity if \( p(\Lambda, BC) \) is negative (positive).

• For lattices with periodic boundary conditions in all directions, \( n(G)/e(G) = 2/\kappa_\Lambda \) and hence

\[ p(\Lambda, \text{tor}) = -\frac{1}{e(G)}. \]
• For the finite lattice graphs with large vertex degree and small girth, e.g., the triangular lattice, another truncation of the partition function when $Q$ is large is

$$Q(v + 1)^{e(G)} + Q^{n(G)^2},$$

so that the center of the circle, denoted by $c(\Lambda, BC)$, is at

$$c(\Lambda, BC) = -\frac{1}{Q^{2/\kappa_\Lambda}},$$

which approaches the origin as $Q \to \infty$.

• For finite lattice graphs with small vertex degree and large girth, e.g., the honeycomb lattice, the other truncation of the partition function when $Q$ is large is

$$Qv^{e(G)} + Q^{n(G) - e(G)}(Q + v)^{e(G)} = Q^{n(G) + 1}[x_G^{e(G)} + Q^{-1}(1 + Q^{(n(G)/e(G)) - 1}x_G^{e(G)})].$$

The circle near which the zeros lie is shifted to the right.

For the lattices with periodic boundary conditions in all directions and $e(G)$ larger than $n(G)$ when $Q$ is large, the radius of the circle is approximately $Q^{-1/e(G)}$ and the center of the circle in the $x_\Lambda$ plane is

$$c(\Lambda, tor) = Q^{-s(\Lambda, BC)},$$

where $s(\Lambda, tor) = 1 + \frac{2 - n(G)}{e(G)}$.

When $e(G)$ is large, so that $2/e(G) \ll 1$, the position of the center can be further approximated as $c(\Lambda, tor) \simeq Q^{(2/\kappa_\Lambda)^{-1}}$, which approaches the origin as $Q \to \infty$. 
6.2.2 Square lattice sections

- Exact calculations of the Potts model partition function on a section of the square lattice of size $L_y = 4$ and $L_x = 9$ with toroidal, cyclic, cylindrical, and free boundary conditions. Consider a typical large value $Q = 1000$.

- For toroidal (tor) boundary conditions, $e(sq, tor) = 2L_xL_y$ and

$$p(sq, tor) = -\frac{1}{2L_xL_y}.$$ 

In the thermodynamic limit, the radius of the approximating circle increases and approaches unity from below.

- For cyclic boundary conditions, $e(sq, cyc) = L_x(2L_y - 1)$ and

$$p(sq, cyc) = \frac{L_x - 2}{2L_x(2L_y - 1)}.$$ 

In the thermodynamic limit, the radius of the approximating circle decreases and approaches unity from above.

- For free boundary conditions, $e(sq, free) = 2L_xL_y - L_x - L_y$ and

$$p(sq, free) = \frac{L_x + L_y - 2}{2(2L_xL_y - L_x - L_y)}.$$ 

Figure 53: Zeros of the Potts model partition function, plotted in the $x_{sq}$ plane, for a section of the square lattice with $L_y = 4$ and $L_x = 9$, when $q = 1000$.

(a) toroidal  
(b) cyclic  

(c) cylindrical  
(d) free
6.2.3 Triangular lattice

- Exact calculations of the Potts model partition function on a section of the triangular lattice of size $L_y = 3$ and $L_x = 9$ with toroidal boundary conditions, and $L_y = 4$ and $L_x = 9$ with cyclic, cylindrical, and free boundary conditions. Consider a typical large value $Q = 1000$.

- For toroidal boundary conditions, $e(\text{tri, tor}) = 3L_xL_y$ and

$$p(\text{tri, tor}) = -\frac{1}{3L_xL_y}.$$ 

In the thermodynamic limit, the radius of the approximating circle increases and approaches unity from below. There is a noticeable shift of the circle to the left. The center of the approximating circle is at $x_{tri} = -Q^{-1/3}$, approaching zero as $Q \to \infty$.

- For cyclic boundary conditions, $e(\text{tri, cyc}) = L_x(3L_y - 2)$ and

$$p(\text{tri, cyc}) = \frac{2L_x - 3}{3L_x(3L_y - 2)}.$$ 

In the thermodynamic limit, the radius of the approximating circle decreases and approaches unity from above.
Figure 54: Zeros of the Potts model partition function in the $x_{tri}$ plane for the triangular lattice when $q = 1000$.

(a) toroidal

(b) cyclic

(c) cylindrical

(d) free
6.2.4 Honeycomb lattice

- Exact calculations of the Potts model partition function on a section of the honeycomb lattice of size $L_y = 4$ and $L_x = 18$ with toroidal and cyclic boundary conditions, and $L_y = 4$ and $L_x = 9$ with cylindrical and free boundary conditions. Consider $Q = 1000$.

- For the toroidal boundary conditions where both $L_x$ and $L_y$ are even, $e(hc, tor) = 3L_xL_y/2$ so that $n(hc, tor)/e(hc, tor) = 2/3$ for any $L_x$ and $L_y$.

  $p(hc, tor) = -\frac{2}{3L_xL_y}, \quad s(hc, tor) = \frac{1}{3} + \frac{4}{3L_xL_y}$.

  In the thermodynamic limit, the radius of the approximating circle increases and approaches unity from below. The center of the approximating circle is close to $x_{hc} = Q^{-1/3}$ as the size of the lattice section becomes large.

- For cyclic boundary conditions, $e(hc, cyc) = L_x(3L_y - 1)/2$, where $L_x$ must be even.

  $p(hc, cyc) = \frac{2(L_x - 3)}{3L_x(3L_y - 1)}, \quad s(hc, cyc) = \frac{1}{3} + \frac{4(3 - L_x)}{3L_x(3L_y - 1)}$.

- For cylindrical boundary conditions, $e(hc, cyl) = L_y(3L_x/2 - 1)$ where $L_y$ must be even.

  $p(hc, cyl) = \frac{2(2L_y - 3)}{3L_y(3L_x - 2)}, \quad s(hc, cyl) = \frac{1}{3} + \frac{4(3 - 2L_y)}{L_y(3L_x - 2)}$. 
Figure 55: Zeros of the Potts model partition function in the $x_{hc}$ plane for the honeycomb lattice when $q = 1000$.

(a) toroidal

(b) cyclic

(c) cylindrical

(d) free
6.2.5 Kagomé lattice

- Exact calculations of the Potts model partition function on a section of the kagomé lattice of size $L_y = 2$ and $L_x = 9$ with toroidal and cylindrical boundary conditions, and $L_y = 3$ and $L_x = 9$ with cyclic and free boundary conditions. Consider a typical large value $Q = 1000$.

- For toroidal boundary conditions, $n(kag, tor) = 3L_xL_y$ and $e(kag, tor) = 6L_xL_y$ so that $n(kag, tor)/e(kag, tor) = 1/2$ as for the square lattice.

$$p(kag, tor) = -\frac{1}{6L_xL_y}.\$$

In the thermodynamic limit, the radius of the approximating circle increases and approaches unity from below.
Figure 56: Zeros in the $x_{kag}$ plane for Potts model partition function on the Kagomé lattice when $q = 1000$.

(a) toroidal

(b) cyclic

(c) cylindrical

(d) free
7 Summary and conclusion

7.1 Potts model partition function on lattice strips

- Exact solutions for Potts model partition function $Z(G, Q, v)$ for arbitrary (not necessarily positive integral) $Q$ and for arbitrary $v$ on lattice strips of arbitrarily great length; resultant exact solutions for the free energy in the limit $L_x \to \infty$.

- Structural results of full-temperature Potts model partition function for cyclic strips and self-dual strips: $Z(G, Q, v) = \sum_j c_{G,j} (\lambda_{G,j})^{L_x}$. Determination of coefficients $c_{G,j}$, numbers of $\lambda_{G,j}$’s for each degree coefficient and their sum.

- Structural results of the chromatic polynomial for cyclic strips and self-dual strips: $P(G, Q) = \sum_j c_{G,j} (\lambda_{G,j})^{L_x}$. Determination of numbers of $\lambda_{G,j}$’s for each degree coefficient and their sum.

- Dimensions of the transfer (coloring) matrices of both full-temperature Potts model partition function and the chromatic polynomial for free and cylindrical strips of the square and triangular lattices.
7.2 Flow polynomial

- We found that the loci $\mathcal{B}_{fl}$ were noncompact for many strip graphs with periodic (or twisted periodic) longitudinal boundary conditions.

- Aside from the trivial case $L_y = 1$, the maximal point, $Q_{cf}$, where $\mathcal{B}$ crosses the real axis, is universal on cyclic and Möbius strips of the square lattice for all widths for which we have calculated it and is equal to the asymptotic value $Q_{cf} = 3$ for the infinite square lattice. This statement is not true for the toroidal and Klein bottle strips.

- Duality relations were used to derive a number of connections between $fl$ and $W$, and $\mathcal{B}_{fl}$ and $\mathcal{B}_W$, for planar families of graphs.

- Of course, for the self-dual families of planar strip, $\mathcal{B}_{fl} = \mathcal{B}_W$ for each value of $L_y$ separately, and in these cases, these loci share the common property of being compact and having $Q_c = Q_{cf} = 3$, the asymptotic value.
7.3 Reliability polynomial

- For a given type of lattice and a given width and choice of transverse boundary conditions, \( r \) and \( B \) are the same for any choice of longitudinal boundary conditions.

- For all of the lattice strip graphs for which we have calculated \( R(G, p) \) and \( B \), we find that this locus consists of arcs (and in some cases a line segment) and does not enclose regions in the complex \( p \) plane. In some simple cases \( B \) is connected, but in general, it may consist of several disjoint components.

- In all of the cases for which we have calculated exact results, whenever \( B \) consists of an arc of a circle, then this circle is centered at \( p = 1 \). More generally, even if \( B \) is not an arc of a circle, it is roughly centered around \( p = 1 \). For families without multiple edges, we find that the component(s) of \( B \) is (are) roughly concave toward the left.

- For certain strips, the zeros of \( R(G, p) \) violate the bound \( |p - 1| \leq 1 \) and the continuous accumulation sets of zeros \( B \) extend outside of the disk \( |p - 1| \leq 1 \). The amounts by which some individual zeros, and the outer part of the loci \( B \), lie outside this disk are small.
7.4 Partition function zeros of a restricted Potts model

7.4.1 Self-dual strips

• For all of the self-dual strips of the square lattice that we have studied, all of the $\bar{\lambda}$ have unit magnitude at $v = -2$, or equivalently $Q = 4$. Hence, all of the curves on $B_v$ meet at the point $v = -2$, and correspondingly, all of the curves on $B_Q$ meet at the point $Q = 4$.

• For all of the self-dual strips of the square lattice that we have studied, $B_Q$ also crosses the interval $1 \leq Q \leq 4$ at

$$Q = Q_{2\ell+1} \quad \text{for } 0 \leq \ell \leq L_y.$$  

Equivalently, $B_v$ crosses the interval $-2 \leq v \leq -1$ at

$$v = v_{2\ell+1} \quad \text{for } 1 \leq \ell \leq L_y.$$  

We conjecture that this holds for arbitrarily large $L_y$. The locus can also cross the real axis at other points.

• For odd $L_y$, the locus $B_Q$ includes the semi-infinite line segment $4 \leq Q \leq \infty$. Equivalently, $B_v$ includes the semi-infinite line segment, $-\infty \leq v \leq -2$.

We conjecture that this holds for arbitrarily large $L_y$. 
7.4.2 Non self-dual strips

- For all of the nontrivial strips, including those of the square, triangular, and honeycomb lattices, that we have studied, we find that $B_v$ passes through $v = -2$ and $B_Q$ passes through $Q = 4$.

- For all of the cyclic (and equivalently, Möbius) strips that we have studied, $B_Q$ also crosses the real axis at

$$Q = Q_{2\ell} \quad \text{for} \quad \ell = 1, \ldots, L_y.$$ 

We conjecture that this holds for arbitrarily large $L_y$. The locus can also cross the real axis at other points.

7.5 Zeros of the Potts model partition function in the large-$Q$ limit

- We have shown a simple truncation of the full partition function suggests that in the thermodynamic limit and the limit $Q \to \infty$ on an arbitrary regular lattice $\Lambda$ of dimensionality $d \geq 2$, the zeros lie on the circle $|x_{\Lambda}| = 1$. (Pearce and Griffiths, 1980; Wu et. al., 1996)

- We have studied the distribution of zeros for finite sections of various two-dimensional lattices for large $Q$, showing how they also lie approximately on circles.
References


