Complex Roots of Chromatic Polynomials

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INI Workshop January 2008
1. Preview

The original stimulus was the work of Tutte (1970) concerning the real roots of chromatic polynomials of planar graphs.

Biggs, Damerell and Sands (1972) studied complex roots of recursive families, such as the cyclic ladders $L_n$:

They found that the chromatic polynomial of $L_n$ is

$$(z^2 - 3z + 3)^n + (z - 1)(3 - z)^n + (z - 1)(1 - z)^n + (z^2 - 3z + 1).$$
They also observed that as $n \to \infty$ the roots cluster around certain curves.

At the time, the form of the polynomial and the behaviour of the roots was unexplained.
A theorem of *Beraha, Kahane and Weiss (1980)* was used by *Read and Royle (1991)* to explain the behaviour of the roots. The polynomial

\[(z^2 - 3z + 3)^n + (z - 1)(3 - z)^n + (z - 1)(1 - z)^n + (z^2 - 3z + 1)\]

has the form

\[\sum \mu_i(z)\lambda_i(z)^n.\]

The limit curves are (parts of) the equimodular curves, defined by

\[|\lambda_i(z)| = |\lambda_j(z)| \quad (i \neq j).\]

In this case

\[|z^2 - 3z + 3| = |z - 3|, \quad |z^2 - 3z + 3| = |z - 1|, \quad |z - 3| = |z - 1|.\]
In 1999 this idea was revisited. A bracelet is constructed by taking \( n \) copies of a graph \( B \) and linking each copy to the next (in cyclic order) with a set of links \( L \).
It can be shown that the number of $k$-colourings of a bracelet can be expressed in the form

$$\text{trace } T(k)^n,$$

where the transfer matrix $T(k)$ is a $(0, 1)$-matrix whose size is a function of $k$.

Replacing $k$ by $z \in \mathbb{C}$ it follows that the chromatic polynomials of bracelets have the form

$$\text{trace } T(z)^n = \sum \mu_i(z) \lambda_i(z)^n,$$

where the $\lambda_i$ are the eigenvalues of $T$ and the $\mu_i$ are their multiplicities.

The Beraha-Kahane-Weiss theorem applies. This means that we must find the equimodular curves associated with the eigenvalues of matrices of the form $T(z)$: that is, curves where $T(z)$ has two eigenvalues with equal modulus.
In some cases it is easy to find the irreducible constituents of $T(z)$ and their eigenvalues.

**Example 1** For the ladder graphs there are three constituents

\begin{align*}
T_0(z) &= (z^2 - 3z + 3), \quad T_1(z) = \begin{pmatrix} 2 - z & 1 \\ 1 & 2 - z \end{pmatrix}, \quad T_2(z) = (1),
\end{align*}

with multiplicities

\begin{align*}
1, & \quad z - 1, & \quad z^2 - 3z + 1.
\end{align*}

The chromatic polynomial is therefore

\begin{align*}
1(z^2 - 3z + 3)^n + (z - 1)((3 - z)^n + (1 - z)^n) + (z^2 - 3z + 1)1^n.
\end{align*}
Biggs, Reinfeld, and Chang studied the generalised dodecahedra, where the components are paths of length 4.

The chromatic polynomial of $D_n$ can be written as

$$
\text{tr}(T_0(z)^n) + (z - 1)\text{tr}(T_1(z)^n) + (z^2 - 3z + 1)\text{tr}(T_2(z)^n) + (z^3 - 5z^2 + 6z - 1).
$$
Here the constituents $T_0$, $T_1$, $T_2$ are small matrices (size 3,6,4) with polynomial entries. Their eigenvalues are *not* polynomials, but it can be shown that the limit curves for $D_n$ are of the following form (roughly!).

We shall explain how the curves are made up, including the reason for the apparent ‘triple points’.
There are several things to do.

2. Formalize the algebra of bracelet colourings.

3. Explain how to compute the equimodular curves, and hence the limit curves.

4. Study the topology of the curves.

5. . . .

6. . . .
2: Algebra of bracelet colourings

This is a brief survey. More details are given in:

Chromatic polynomials and representations of the symmetric group


Specht modules and chromatic polynomials

We begin with the transfer matrix for bracelets.

For simplicity, we shall take the components $B$ to be complete graphs $K_b$. Two $k$-colourings $\alpha, \beta$ of $K_b$ are $L$-compatible if $\alpha(v) \neq \beta(w)$, for all $(v, w) \in L$. This means that the graph formed by two copies of $K_b$ linked by $L$ is properly coloured by colouring the $K_b$'s as in $\alpha$ and $\beta$.

Define the transfer matrix $T_L$ by

$$(T_L)_{\alpha \beta} = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are } L\text{-compatible;} \\ 0 & \text{otherwise.} \end{cases}$$

So, for each $k$, $T_L$ is a $(0,1)$-matrix whose size is $k(k - 1) \cdots (k - b + 1)$. 
Theorem  The number of $k$-colourings of a bracelet with $n$ components is the trace of $(T_L)^n$.

Proof  Write $T = T_L$. Then

$$\text{trace } T^n = \sum_{\alpha} (T^n)_{\alpha\alpha} = \sum_{\alpha} \sum_{\beta, \gamma, \ldots, \omega} T_{\alpha\beta}T_{\beta\gamma} \cdots T_{\omega\alpha}.$$ 

Now $T_{\alpha\beta}T_{\beta\gamma} \cdots T_{\omega\alpha}$ is nonzero (and equal to 1) if and only if the colourings $\alpha, \beta, \gamma, \ldots, \omega$ unite to form a proper colouring of the whole bracelet.

Because the names of the $k$ colours are irrelevant, it turns out that $T$ can be represented by constituent matrices $T_s(k)$ with fixed size, but with entries and multiplicities that are polynomial functions of $k$. 
For bracelets with $K_b$ components we can give general rules.

For any $k \geq b$, a permutation $\omega \in Sym_k$ acts on the $k$-colourings of $K_b$ in the obvious way: $\alpha \mapsto \omega \alpha$.

There is a corresponding matrix $R(\omega)$ operating on the space of $k$-colourings:

$$R(\omega)_{\alpha \beta} = \begin{cases} 1 & \text{if } \beta = \omega \alpha; \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma** For any $\omega \in Sym_k$, $T_L R(\omega) = R(\omega) T_L$. 
The reduction of $R(\omega)$ to its irreducible constituents can be expressed in the standard theory of representations of the symmetric group (Specht modules). Essentially, $Sym_k$ is acting on the set of injections of a $b$-set into a $k$-set.

Another standard result of representation theory (Schur’s Lemma) gives the relationship between the constituents of $T_L$ and those of $R(\omega)$.

The outcome is a simple description of the multiplicity and size of the constituents $T_{L,\pi}$ of $T_L$, corresponding to integer partitions $\pi$. 
**Notation**  When \( \pi \) is a partition of the integer \( \ell \) we write \(|\pi| = \ell\) and denote the parts by \( \pi_1 \geq \pi_2 \geq \cdots \geq \pi_\ell \), where *any part may be zero*. Put \( \sigma_i = \pi_i + \ell - i \) and define

\[
d_\pi = \frac{\ell! \times \prod (\sigma_i - \sigma_j)}{\prod \sigma_i!}.
\]

Each \( \pi \) such that \(|\pi| = \ell\) determines an irreducible representation \( S^\pi \) of the symmetric group \( Sym_\ell \), with dimension \( d_\pi \). These form a complete set. For example, when \( \ell = 3 \) the irreducible representations are

\[
S^{[300]} \quad S^{[210]} \quad S^{[111]}
\]

with dimensions

\[
1 \quad 2 \quad 1.
\]
Theorem  The number of $k$-colourings of a bracelet $B_n$ with components $K_b$ and links $L$ can be written as a sum of terms, one for each partition $\pi$ with $0 \leq |\pi| \leq b$:

$$\sum_{\pi} m_\pi(k) \text{trace}(T_{L,\pi})^n.$$ 

Here

$$m_\pi(k) = f_\pi (k - \sigma_1)(k - \sigma_2) \cdots (k - \sigma_{|\pi|}), \quad \text{where}$$

$$f_\pi = d_\pi/|\pi|!, \quad \sigma_i = |\pi| + \pi_i - i,$$

and $T_{L,\pi}$ is a square matrix in which the entries are polynomials in $k$ that depend on $L$, and the size is $(b_{|\pi|})d_\pi$, independent of $L$. 

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Example Suppose the components are $K_3$’s and the links join corresponding vertices.
Here $b = 3$ and the multiplicities are:

$$m_{[0]}(k) = 1 \quad m_{[1]}(k) = k - 1$$

$$m_{[20]}(k) = \frac{1}{2}(k - 3)k \quad m_{[11]}(k) = \frac{1}{2}(k - 2)(k - 1)$$

$$m_{[300]}(k) = \frac{1}{6}(k - 5)(k - 1)k$$

$$m_{[210]}(k) = \frac{2}{6}(k - 4)(k - 2)k$$

$$m_{[111]}(k) = \frac{1}{6}(k - 3)(k - 2)(k - 1).$$

Replacing $k$ by $z$, and calculating the eigenvalues of the matrices $T_{L,\pi}$ when $L$ is the given set of links, we get the chromatic polynomial originally found by Biggs and Shrock 1999:
\[ \pi = [0] : \quad 1(z^3 - 6z^2 + 14z - 13)^n \]

\[ \pi = [1] : \quad +(z - 1)\{(\neg z^2 + 7z - 13)^n + 2(\neg z^2 + 4z - 4)^n\} \]

\[ \pi = [20] : \quad +\frac{1}{2}(z^2 - 3z)\{(z - 5)^n + 2(z - 2)^n\} \]

\[ \pi = [11] : \quad +\frac{1}{2}(z^2 - 3z + 2)\{2(z-4)^n + (z-1)^n\} \]

\[ \pi = [300], [210], [111] : \quad +(z^3 - 6z^2 + 8z - 1)(-1)^n. \]

In the last line the multiplicity is the sum of the terms corresponding to the partitions of 3.

In this example the eigenvalues of the constituents are polynomials. The eigenvalues depend on the links, but the multiplicities would be the same for any bracelet with \( K_3 \) components.
When the components of the bracelet are not complete, we look at each class of $k$-colourings separately, and ‘collapse’ accordingly. For example, when the components are paths of length 4, (as for the generalized dodecahedra) there are five classes:
3: Equimodular curves and limit curves

For more details, see:


Also three research reports:

*LSE-CDAM-2000-07, LSE-CDAM-2000-17, LSE-CDAM-2001-01*

available via the LSE Mathematics Department website.
The Story So Far

• The chromatic polynomial of a bracelet with \( n \) components can be written in the form

\[
\sum \mu_i(z) \lambda_i(z)^n,
\]

where the \( \lambda_i(z) \) are eigenvalues of certain matrices.

• As \( n \to \infty \) the roots of such a family of polynomials approach parts of the equimodular curves defined by

\[
|\lambda_i(z)| = |\lambda_j(z)|.
\]

We must explain what is meant by parts of.
Let us say that an eigenvalue $\lambda_m(z)$ is **dominant** at $z_0$ if $|\lambda_m(z_0)| \geq |\lambda_i(z_0)|$ for all eigenvalues $\lambda_i(z)$.

Consider an equimodular curve $|\lambda_a(z)| = |\lambda_b(z)|$. If a part of the curve is the boundary between a region in which $\lambda_a(z)$ is dominant and a region in which $\lambda_b(z)$ is dominant, we shall say that this is a **dominant part** of the curve.

The Beraha-Kahane-Weiss theorem says that only dominant parts of the equimodular curves can be limit points of zeros.
Suppose an equimodular curve $|\lambda_a(z)| = |\lambda_b(z)|$ intersects another equimodular curve $|\lambda_a(z)| = |\lambda_c(z)|$ at $z_1$. 
Suppose an equimodular curve $|\lambda_a(z)| = |\lambda_b(z)|$ intersects another equimodular curve $|\lambda_a(z)| = |\lambda_c(z)|$ at $z_1$.

Then clearly the equimodular curve $|\lambda_b(z)| = |\lambda_c(z)|$ also passes through $z_1$. 
If the three eigenvalues are not dominant at $z_1$ then the curves have no dominant parts in a neighbourhood of $z_1$.

But if they are dominant, the configuration in the neighbourhood of $z_1$ must be as follows, where the bold lines indicate dominant parts.

We shall call $z_1$ a triplet point. Note that (in general) a triplet point is not a singularity of any of the three equimodular curves, although the limit curves will appear to be singular.
Example  For the ‘toroidal’ graphs with $B = K_3$ the eigenvalues are

$$\lambda_a = z^3 - 6z^2 + 14z - 13, \quad \lambda_b = -z^2 + 7z - 13,$$

$$\lambda_c = -z^2 + 4z - 4, \quad \lambda_d = z - 5, \quad \lambda_e = z - 2,$$

$$\lambda_f = z - 4, \quad \lambda_g = z - 1 \quad \lambda_h = -1.$$

We can easily obtain the equations of the equimodular curves in $(x, y)$ coordinates. For example, the equation $|\lambda_b| = |\lambda_d|$ is

$$y^4 + 2(x^2 - 7x + 11)y^2 + (x - 2)(x - 4)(x^2 - 8x + 18) = 0.$$

Consider first the three curves arising from $\lambda_a, \lambda_b, \lambda_c$. 
There are no intersections. It follows that $\lambda_c$ never dominates, being dominated by $\lambda_a$ outside the first curve and $\lambda_b$ inside it.

Now consider $\lambda_a, \lambda_b, \lambda_d$. 
There are triplet points at $(5 \pm 2\sqrt{2}i)/2$. Removing the non-dominant parts of the curves, we get the limit curves as follows. (It can be checked that the remaining eigenvalues do not contribute.)
The general situation is more complicated because, although the terms $\lambda_i(z)$ are eigenvalues of a square matrix $A(z)$ over $\mathbb{Z}[z]$, they are not themselves polynomial functions.

So we must consider the characteristic polynomial of $A(z)$

$$\text{det}(\lambda I - A(z)) = \lambda^m + a_1(z)\lambda^{m-1} + \cdots + a_m(z),$$

which has coefficients in $\mathbb{Z}[z]$.

We need to find points $z \in \mathbb{C}$ where this equation has two roots $\lambda, \lambda'$ with equal modulus.
For the moment, treat the coefficients as indeterminates $a_1, a_2, \ldots, a_m$.

The basic idea (Salas and Sokal) is that if $|\lambda'| = |\lambda|$, then $\lambda' = s\lambda$ for some $s$ with $|s| = 1$. In other words we need to find points where the equations

$$\lambda^m + a_1\lambda^{m-1} + \cdots + a_m = 0$$

$$s^m\lambda^m + a_1s^{m-1}\lambda^{m-1} + \cdots + a_m = 0$$

have common root.

There is a classical condition for this – the vanishing of a determinant known as the resultant of the two equations.
For example, for an equation $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ of degree $m = 3$, the resultant is

\[
\begin{vmatrix}
1 & a_1 & a_2 & a_3 & 0 & 0 \\
0 & 1 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & 1 & a_1 & a_2 & a_3 \\
s^3 & s^2a_1 & sa_2 & a_3 & 0 & 0 \\
0 & s^3 & s^2a_1 & sa_2 & a_3 & 0 \\
0 & 0 & s^3 & s^2a_1 & sa_2 & a_3
\end{vmatrix}
\]

For general $m$, this determinant can be reduced to the form

\[a_m(s - 1)^m P_m(s),\]

where $P_m$ is a \textit{reciprocal} polynomial of degree $m(m - 1)$. 
Consequently it is convenient to change the variable from $s$ to
\[ t = s + s^{-1} + 2, \quad \text{that is} \quad s = e^{i\theta}, \; t = 4 \cos^2(\theta/2). \]
So the fact that $|s| = 1$ corresponds to $t$ being in the real interval $[0, 4]$.

Making this substitution in the case $m = 3$, and removing factors, we conclude that $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ has two roots with equal modulus when there is a $t \in [0, 4]$ such that $r(t, a_1, a_2, a_3) = 0$, where

\[
 r(t, a_1, a_2, a_3) =
 (t - 1)^3 a_3^2 - (t - 1)(t + 2)a_1a_2a_3 + ta_2^3 + ta_1^3 a_3 - a_1^2 a_2^2.
\]
Suppose $A(z)$ is a matrix with entries in $\mathbb{Z}[z]$, so that its characteristic polynomial also has coefficients $a_i(z) \in \mathbb{Z}[z]$.

Replacing the indeterminates $a_i$ in $r(t, a_1, \ldots, a_m)$ by $a_i(z)$, we get a function $v_A(t, z)$, polynomial in both variables and with integer coefficients, with the following property.

**Theorem** $A(z)$ has two eigenvalues with equal modulus if and only if

$$v_A(t, z) = 0$$

for some $t \in [0, 4]$. 
4: Topology of the curves

The condition $v_A(t, z) = 0$ provides an *implicit* definition of the equimodular curves associated with $A(z)$.

Specifically, suppose $v_A(t_0, z_0) = 0$. If the mapping $z \mapsto v_A(t_0, z)$ is not singular at $z_0$ (that is, if the Jacobian is not zero) there is a neighbourhood of $N$ of $t_0$ and a continuously differentiable function $\zeta$ defined on $N$ such that $\zeta(t_0) = z_0$ and $v_A(t, \zeta(t))$ is identically zero in $N$. In the absence of singularities, we get a smooth arc $\zeta : [0, 4] \rightarrow \mathbb{C}$. 

![Diagram](image)
$v_A$ can be written as a polynomial in $z$ with coefficients in $\mathbb{Z}[t]$:

$$v_A(t, z) = c_0(t)z^r + c_1(t)z^{r-1} + \cdots + c_r(t).$$

**Lemma** The Jacobian of $z \mapsto v_A(t, z)$ is the discriminant of this polynomial in $z$.

The Jacobian is therefore a function $J_A(t)$ depending on the coefficients $c_0(t), c_1(t), \ldots, c_r(t)$. The roots of $J_A(t) = 0$ provide information about the equimodular curves, in particular the endpoints ($t = 0$ and $t = 4$), and the singularities ($t = t^*$, $0 < t^* < 4$).
• All roots of $v_A(0, z)$ are double roots. So $J_A(t)$ has a factor $t^a$ ($a \geq 1$), and there are $2a$ arcs linking up in pairs.

• If $v_A(4, z) = 0$ has $b$ double roots, then $J_A(t)$ has a factor $(t - 4)^b$. The arcs link up here too, and may form *closed* curves.

• If $J_A(t^*) = 0$ with $0 < t^* < 4$, then the curve has an ‘internal’ singularity.
Here are some examples based on $2 \times 2$ matrices.

For a general $2 \times 2$ matrix

$$
\begin{pmatrix}
  n(z) & p(z) \\
  q(z) & r(z)
\end{pmatrix},
$$

the characteristic equation is

$$
\lambda^2 + a_1(z)\lambda + a_2(z) = 0,
$$

where $a_1(z) = -n(z) - r(z)$, $a_2(z) = n(z)r(z) + q(z)p(z)$.

Calculating the resultant of $\lambda^2 + a_1\lambda + a_2$ and $s^2\lambda^2 + sa_1\lambda + a_2$, putting $t = s + s^{-1} + 2$, and removing factors, gives

$$
v(t, z) = ta_2(z) - a_1(z)^2.
$$
Example 1 Take

\[
A(z) = \begin{pmatrix} 4z & 2z - 1 \\ 2 & 2z \end{pmatrix}, \quad \text{for which}
\]

\[
v_A(t, z) = (8t - 36)z^2 - 4tz + 2t,
\]

\[
J_A(t) = 16t(3t - 18).
\]


\(J_A(t)\) vanishes (once) when \(t = 0\) as expected, so there are two arcs forming a single segment. It also vanishes at \(t = 6\), which is not in \([0, 4]\), so there are no internal singularities.

The endpoints are given by \(v_A(0, z) = 0\) and \(v_A(4, z) = 0\):

\[
t = 0 : \quad z = 0 \text{ (twice)}
\]

\[
t = 4 : \quad z = -2 \pm \sqrt{6}.
\]
In this case it is easy to check directly that when $z$ is a *real* number in the interval $(-2 - \sqrt{6}, -2 + \sqrt{6})$, the matrix $A(z)$ has two complex conjugate eigenvalues, which trivially have equal modulus.

Thus the equimodular curve comprises two arcs lying on the real axis.
Example 2 Take

\[ B(z) = \begin{pmatrix} 4z & 2z + 1 \\ 2 & 2z \end{pmatrix}, \quad \text{for which} \]

\[ v_B(t, z) = (8t - 36)z^2 - 4tz - 2t, \]

\[ J_B(t) = 16t(5t - 18). \]

Here \( J_B(t) \) vanishes when \( t = 0 \), and when \( t = 18/5 \), which is in \([0, 4]\).

The endpoints are:

\[ t = 0 : \quad z = 0 \text{ (twice)} \]

\[ t = 4 : \quad z = -2 \pm \sqrt{2}, \]

and the singularity is

\[ t = 18/5 : \quad z = -1 \text{ (twice)}. \]
So we have the following confusing clues.

As in the previous example, we can check directly that $B(z)$ has two complex conjugate eigenvalues (hence with equal modulus) when $z$ is a real number in the interval $(-2 - \sqrt{2}, -2 + \sqrt{2})$. But how to reconcile this with the known endpoints and the singularity?
So we have the following confusing clues.

As in the previous example, we can check directly that $B(z)$ has two complex conjugate eigenvalues (hence with equal modulus) when $z$ is a real number in the interval $(-2 - \sqrt{2}, -2 + \sqrt{2})$.

But how to reconcile this with the known endpoints and the singularity? In fact the equimodular curve and its decomposition into arcs is as follows.
Example 3 Take

\[ C(z) = \begin{pmatrix} 4z & 2z + 2 \\ 2 & 2z \end{pmatrix}, \quad \text{for which} \]

\[ v_C(t, z) = (8t - 36)z^2 - 4tz - 4t, \]

\[ J_C(t) = 16t(9t - 36). \]

Since \( J_C(t) = 0 \) when \( t = 0 \) and \( t = 4 \), both endpoints of the two arcs coincide, and the curve is closed.
In this case \( C(z) \) has eigenvalues \( \lambda_1(z) = 4z + 2 \) and \( \lambda_2(z) = 2z - 2 \), so the matrix is reducible over \( \mathbb{Z}(z) \).

In fact, there is a general theorem.

**Theorem** Given a matrix of the form

\[
\begin{pmatrix}
U(z) & V(z) \\
O & W(z)
\end{pmatrix}
\]

over \( \mathbb{Z}(z) \), the equimodular curves defined by one eigenvalue of \( U(z) \) and one eigenvalue of \( W(z) \) are closed.

Consequently, if all the eigenvalues are polynomials, then all the curves are closed.
Example 4 Take

\[ D(z) = \begin{pmatrix} 4z & 2z + 3 \\ 2 & 2z \end{pmatrix}, \text{ for which} \]

\[ v_D(t, z) = (8t - 36)z^2 - 4tz - 6t, \]

\[ J_D(t) = 16t(13t - 54). \]

Here \( t = 0 \) is the only root of \( J_D(t) = 0 \) in the interval \([0, 4]\), so there are no singularities, and the curve is as follows.
A real-life example – the family of generalized dodecahedra. Recall that the chromatic polynomial of $D_n$ is

$$\text{tr} T_0(z)^n + (z - 1)\text{tr} T_1(z)^n + (z^2 - 3z + 1)\text{tr} T_2(z)^n + (z^3 - 5z^2 + 6z - 1),$$

where $T_0(z), T_1(z), T_2(z)$ are matrices of size 3, 6, 4. This formula leads to $(13 \times 12)/2 = 78$ equimodular curves. Here we shall look only at the curves defined by eigenvalues of $T_0(z)$.

The coefficients of the characteristic polynomial of $T_0(z)$ are (putting $w = z - 2$)

$$a_1 = -(w^4 + 2w^3 + 4w^2 + 1),$$
$$a_2 = w(w + 1)(w^4 + w^3 + 2w^2 + 2),$$
$$a_3 = -w^2(w + 1)^2.$$
Substituting these polynomials in the generic expression $v(t, a_1, a_2, a_3)$ for a $3 \times 3$ matrix, writing the result as a polynomial in $w = z - 2$ with $\mathbb{Z}[t]$ coefficients, and removing unimportant factors, leads to

$$v(t, z) = w^{16} + 6w^{15} + (25 - t)w^{14} + \cdots + (2t^2 - 6t - 4)w + (4 - t).$$

The endpoints of the arcs are obtained by solving the equations $v(0, z) = 0$ and $v(4, z) = 0$, both of degree 16. The first equation has 8 double roots and the second has 3 double roots and 10 single roots.
The equimodular curves comprise 16 arcs, joined up in 8 pairs, thus:

The 8 roots of $v(0, z) = 0$ and the 13 roots of $v(4, z) = 0$ are located as follows.

There may also be some singularities.
The Jacobian (discriminant of $v(t, z)$ as a polynomial in $z$) is
\[ J(t) = K t^8 (t - 4)^3 F(t), \]
where $F(t)$ is a polynomial of degree 48.

There are two roots of $F(t) = 0$ in the interval $(0, 4)$
\[ t_1 = 2.358\ldots, \quad t_2 = 3.085\ldots. \]
These give rise to singularities at $z_1 = 1.466\ldots$ and $z_2 = 2.337\ldots$.

The equimodular curves are as follows.
It remains only(!) to identify the dominant parts, and include the curves arising from the other constituent matrices.
5. A Fully Polynomial Family?

We consider the family of bracelets $G_{r,n}$ with $B = K_r$, a complete graph, and $L = I$, the identity matching. For example, when $r = 3$:

The aim is to describe what happens when both $n$ and $r$ are large.
We recall the case of the general theorem when $L = I$.

**Theorem** The chromatic polynomial of the bracelet $G_{r,n}$ with $n$ components $K_r$ and links $I$ can be written as a sum of terms, one for each partition $\pi$ with $0 \leq |\pi| \leq r$:

$$\sum_\pi m_\pi(z) \text{trace}(T_{I,\pi})^n.$$ 

Here

$$m_\pi(z) = \frac{d_\pi}{|\pi|!} (z - \sigma_1)(z - \sigma_2) \cdots (z - \sigma_{|\pi|}),$$

where $d_\pi$ is the dimension of the representation associated with $\pi$ and $\sigma_i = |\pi| + \pi_i - i$.

$T_{I,\pi}$ is a square matrix, with size $\binom{r}{|\pi|}d_\pi$, in which the entries are polynomials in $z$. 

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**Conjecture** For this family, all eigenvalues are polynomials.

Specifically: For each $\pi$ with $0 \leq |\pi| \leq r$, all the eigenvalues of the matrix $T_\pi = T_{I,\pi}$ are polynomials of degree $r - |\pi|$. We usually say that partitions with $|\pi| = \ell$ are at *level* $\ell$.

The conjecture has been verified for some $|\pi|$, by constructing $T_\pi$. For example:

*Level 0:* $T_{[0]}$ is a $1 \times 1$ matrix with eigenvalue $a_r(z)$ of degree $r$.

*Level 1:* $T_{[1]}$ is an $r \times r$ matrix with two distinct eigenvalues, both polynomials of degree $r - 1$. Their multiplicities are 1 and $r - 1$. 
**Level 2:**

$T_{[20]}$ is a $\binom{r}{2} \times \binom{r}{2}$ matrix with three distinct eigenvalues, polynomials of degree $r - 2$. Their multiplicities are 1, $r - 1$, and $r(r - 3)/2$.

$T_{[11]}$ is a $\binom{r}{2} \times \binom{r}{2}$ matrix with two distinct eigenvalues, polynomials of degree $r - 2$. Their multiplicities are $r - 1$ and $(r - 1)(r - 2)/2$.

In fact all the polynomials mentioned above correspond to the one-dimensional (principal and alternating) representations, for which explicit formulae are known (see below).
Level 3: Apart from $T_{[300]}$ and $T_{[111]}$ (the principal and alternating representations, for which the eigenvalues are known), there is only $T_{[210]}$ to consider.

Sub-Conjecture 1: When $r \geq 5$, $T_{[210]}$ has four distinct eigenvalues, polynomials of degree $r - 3$.

I don’t have a sensible conjecture for the multiplicities.
Level $\ell$: In general, only the eigenvalues of $T_\pi$ for the principal and alternating representations are known:

$\pi = [\ell00 \cdots 0]$ (principal representation)  
Let $m = \min(\ell, r - \ell)$. Then there are $m + 1$ eigenvalues of degree $r - \ell$:

$$
p_{r0}^\ell(z) \quad p_{r1}^\ell(z) \quad p_{r2}^\ell(z) \quad \ldots \quad p_{rm}^\ell(z)
$$

multiplicities: $1 \quad r - 1 \quad \binom{r}{2} - \binom{r}{1} \quad \ldots \quad \binom{r}{m} - \binom{r}{m-1}$.

$\pi = [111 \cdots 1]$ (alternating representation)  
When $\ell \leq r - 1$ there are two eigenvalues of degree $r - \ell$:

$$
q_{r0}(z) \quad q_{r1}(z)
$$

multiplicities: $\binom{r-1}{\ell-1} \quad \binom{r-1}{\ell}$.
Sub-Conjecture 2 For every partition $\pi$ of $r - 1$, every eigenvalue of $T_\pi$ is of the form $(-1)^{r-1}(z - i)\ (1 \leq i \leq 2r - 1)$, and (for $r \geq 4$) every such polynomial occurs.

The results for the principal and alternating representations imply that

$\pi = [r - 1\ 00\ldots0]$ contributes the terms $z - 2r + 1$ (mult 1) and $z - r + 1$ (mult $r - 1$).

$\pi = [111\ldots1]$ contributes the terms $z - 1$ (mult 1) and $z - r - 1$ (mult $r - 1$).

Sub-Conjecture 2.1 If $\pi$ contributes the term $z - i$ then $\pi^*$ contributes the term $z - i^*$ (with the same multiplicity). Here $\pi^*$ is the conjugate of $\pi$ and $i^* = 2r - i$. 
Level $r$: Theorem For every partition of $r$, $T_\pi = (-1)^r I$, where $I$ is the identity matrix of size $d_\pi$. Hence all the eigenvalues are $(-1)^r$.

For an attempt to find more eigenvalues, see: Chromatic Polynomials of some Families of Graphs I: Theorems and Conjectures, CDAM Research Report LSE-CDAM-2005-09.
Fix $r$

Recall that $T_{[0]}$ is a $1 \times 1$ matrix: $T_{[0]} = (a_r(z))$, where $a_r(z)$ is a monic polynomial of degree $r$, which is also the eigenvalue. In fact, for a positive integer $k$, $a_r(k)$ is the number of $k$-colourings of $K_r$ that are compatible with a given one.

This observation leads to a formula for $a_r(z)$. For $z \in \mathbb{C}$ define

$$(z)_0 = 1, \quad (z)_i = z(z - 1)(z - 2) \cdots (z - i + 1).$$

Then

$$a_r(z) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (z-j)_{r-j}.$$

For example $a_0(z) = 1$, $a_1(z) = z - 1$, $a_2(z) = z^2 - 3z + 3$, $a_3(z) = z^3 - 6z + 14z - 13$. 
Define

\[ A_j(z) = \begin{cases} a_j(z - r + j) & \text{if } j = 0, 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases} \]

**Theorem** Let \( m = \min(\ell, r - \ell) \). The eigenvalues of \( T[\ell \ldots 0] \) are \( p_{ri}(z) \) (0 \( \leq \) \( i \) \( \leq \) \( m \)), where

\[
p_{ri}(z) = \sum_{j=r-2\ell}^{r-\ell} \omega_{ij} A_j(z).
\]

The coefficients \( \omega_{ij} \) are integers; explicitly

\[
\omega_{ij} = (-1)^{r+j} (r - \ell - j)! E_{r-\ell-j}(i),
\]

where \( E_s(i) = \) formula given in LAA (2002) p.18.
The result for the two eigenvalues of $T_{[11\ldots1]}$ is simpler.

$$q_{r0}(z) = (-1)^\ell (A_{r-\ell}(z) - (r - \ell)A_{r-\ell-1}(z)),$$

$$q_{r1}(z) = (-1)^\ell (A_{r-\ell}(z) + \ell A_{r-\ell-1}(z)).$$

We shall say that the eigenvalues of the matrices $T_{[\ell0\ldots0]}$ for $0 \leq \ell \leq r$ are the **principal eigenvalues**, and the eigenvalues of $T_{[11\ldots1]}$ for $0 \leq \ell \leq r$ are **alternating eigenvalues**. At each level $\ell \geq 2$ there are

$$\min(\ell, r - \ell) + 1$$

principal eigenvalues and two alternating eigenvalues, and they are polynomials of degree $r - \ell$. 
Example  Using the formulae given above, and an *ad hoc* calculation for [210], we can list all the eigenvalues (and their multiplicities) when $r = 4$.

\[
egin{align*}
z^4 & - 10z^3 + 41z^2 - 84z + 73 \\
- z^3 & + 12z^2 - 50z + 73 \\
z^2 & - 11z + 31 \\
z^2 & - 9z + 21 \\
- z & + 7 \\
- z & + 5 \\
- z & + 6 \\
1
\end{align*}
\]

Chang extended this to $r = 5$ and $r = 6$. 
We now look at the equimodular curves and the limit curves.

**Wild Conjecture** For $r \geq 3$, the only eigenvalues that contribute to the limit curves are the principal and alternating eigenvalues.

In other words, at every $z \in \mathbb{C}$, every eigenvalue of every $T_\pi$ ($0 \leq |\pi| \leq r$) is dominated by one of the eigenvalues of $T[\ell\,0\ldots\,0]$, $T[1\ldots\,1]$ for some $0 \leq \ell \leq r$.

There is very little justification for the Wild Conjecture, but if true it would reduce the problem of finding the limit curves to a manageable size.
**Convention**  
Eigenvalues at level \( \ell \) have the form \((-1)^\ell \lambda(z)\), where \( \lambda(z) \) is a monic polynomial. Henceforth we shall refer to \( \lambda(z) \) as the eigenvalue.

At level \( r \) there is one eigenvalue 1, for all \( \pi \) with \( |\pi| = r \).

At level \( r - 1 \) there are two principal eigenvalues \( z - 2r + 1 \) and \( z - r + 1 \), and two alternating eigenvalues \( z - 1 \) and \( z - r - 1 \).

At every point \( z \), one of \( |z - 2r + 1| \), \( |z - 1| \) is greater than \( |1| \), so level \( r \) can be disregarded.

What about level \( r - 1 \)?
We have the geometrically obvious

**Lemma**  If \( z = x + iy \) and \( p, q \in \mathbb{R} \) with \( p > q \), then
\[
|z - p| > |z - q| \text{ if and only if } x < \frac{(p + q)}{2}.
\]

**Corollary**  Among the principal and alternating eigenvalues at level \( r - 1 \), \( z - 2r + 1 \) dominates in the region \( x < r \), and \( z - 1 \) dominates in the region \( x > r \).

If all eigenvalues at level \( r - 1 \) are of the form \( z - p \) with \( 1 \leq p \leq 2r - 1 \), (Sub-Conjecture 2), this result accounts for all eigenvalues at level \( r - 1 \).
At level $r - 2$ there are three principal eigenvalues

\[ u_r(z) = z^2 - (4r - 5)z + (4r^2 - 10r + 7), \]

\[ v_r(z) = z^2 - (3r - 5)z + (2r^2 - 7r + 7), \]

\[ w_r(z) = z^2 - (2r - 3)z + (r^2 - 3r + 3), \]

and two alternating eigenvalues

\[ z^2 - (2r + 1)z + (r^2 - r + 1), \]

\[ z^2 - (r + 1)z + (r + 1). \]

We have to compare the moduli of these among themselves and with the moduli of the dominant eigenvalues at level $r - 1$, that is $z - 2r + 1$ and $z - 1$. 
To get started, some algebra is feasible. For example, the equimodular curve for $u_r$ and $v_r$ is, in $(x, y)$ coordinates:

$$(3r - 4 - 2x)y^2 = (x - 2r + 3)Q_r(x),$$

where $Q_r(x)$ is a quadratic. For $r \geq 6$ the roots of $Q_r(x)$ are real numbers $\sigma_r, \tau_r$, and as $r \to \infty$

$$\sigma_r \sim (3r - 4)/2 \text{ from above, } \tau_r \sim 2r - 3 \text{ from below.}$$

So the curve has two components, which collapse towards the asymptote $x = (3r - 4)/2$ and the point $(2r - 3, 0)$.
In fact the equimodular curves for the three principal eigenvalues $u_r, v_r, w_r$ are as follows.

Hence except possibly for small $r$ the (relatively) dominant eigenvalues are the first and third, with boundary at $x = (3r - 4)/2$. 

Hence **except possibly for small** $r$ the (relatively) dominant eigenvalues are the first and third, with boundary at $x = (3r - 4)/2$. 
