Integer Symmetric Matrices with Spectral Radius $\leq 2.019$

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Polynomials we consider:

\[ x_A^n(x) = \det(xI - A) = \prod_{i=1}^{n} (x - \lambda_i) \]

A integer symmetric matrix

\[ \text{e.g., } A = A_G = \text{adjacency matrix of graph } G. \]

Spectral radius \( R_A = \max \lambda_i \)

\( * \) \( R_A \leq 2 \)

A cyclotomic matrix

\( 3^n x_A(3 + \frac{1}{3}) \) is a cyclotomic polynomial

\( 2 \leq R_A < 2.019 \)

\( 1 \)

Find all cyclotomic matrices!

\( * \)

Simplifying the Problem:

I. Decomposable matrices

Defn. A is decomposable if by simultaneously reordering its rows & columns it becomes

\[
\begin{pmatrix}
\square & \square \\
\square & \square
\end{pmatrix}
\]

\(\geq 2\) blocks)

Otherwise it is indecomposable.

1st simplification:

Can assume that A is indecomposable.
Simplifying the Problem:

II Matrix equivalence.

A cyclotomic \[ \lambda \in \mathbb{Z}_n \] \[ | \lambda | \leq 2 \]

Then we can

- replace \( A \) by \( -A \)
- For some \( i \), multiply row \( i \) & column \( i \) by \( -1 \).
- Simultaneously reorder its rows & columns.

Then new matrix is again cyclotomic.

These operations generate an equivalence relation on set of cyclotomic matrices.

Enough to find equivalence class representative.
Interlude: Cauchy Interlacing

Thm (Cauchy). A symmetric, eigenvalue
\[ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \]
\[ \mathbf{A}' = \mathbf{A} \text{ with row } i \text{ and column } i \text{ removed}, \text{ eigenvalues} \]
\[ \lambda_1' \leq \lambda_2' \leq \ldots \leq \lambda_{n-1}' \]

Then
\[ \lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \lambda_2' \leq \ldots \leq \lambda_{n-1} \leq \lambda_{n-1}' \leq \lambda_n \]

Apply this to cyclotomic matrices.
Simplifying the Problem:

III  Maximal cyclotomic matrices

**Defn** A cyclotomic matrix $A'$ is non-maximal if $A'$ can be obtained by deleting the $i$th row & column from some cyclotomic matrix $A$.

Otherwise it is maximal.

It turns out that we can restrict our attention to maximal cyclotomic, because "Every non-maximal cyclotomic matrix is contained in a maximal one".
From simplifications I, II, III:

It's enough to look for equivalence class representatives of indecomposable, maximal cyclotomic matrices.
Known Results: Graphs

- **Graph** ↔ \( \{0,1\} \) - symmetric matrix with 0s on diagonal, "Adjacency matrix"

- Connected ↔ Indecomposable

- Vertices not labelled ↔ Equivalence class

- Cyclotomic graph ↔ Cyclotomic matrix

**Example**

\[ G' = \begin{array}{ccc}
\cdot & \cdot & \\
\cdot & \cdot & \\
\end{array} \quad \iff \quad A' = \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix} \]

- Non-maximal cyclotomic

\[ G = \begin{array}{ccc}
\cdot & \cdot & \\
\cdot & \cdot & \\
\end{array} \quad \iff \quad A = \begin{pmatrix} 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \end{pmatrix} \]

- Maximal cyclotomic
Maximal connected cyclotomic graphs

Theorem (J.H. Smith 1970). These are $\hat{E}_6, \hat{E}_7, \hat{E}_8, \tilde{A}_n \ (n \geq 2), \ \tilde{D}_n \ (n \geq 4)$.

We now extend Smith's results to general integer symmetric matrices.
Figure 9. The maximal connected cyclotomic graphs $E_6, E_7, E_8, A_n, D_n, (n \geq 2)$. The number of vertices is 1 more than the index. (From [MS]).
Step 1: Matrices containing an entry \( \geq 2 \) in modulus.

Proposition. Every indecomposable maximal cyclotomic matrix having an entry \( \geq 2 \) in modulus is equivalent either to the 

- 1x1 matrix (2)
- 2x2 matrix \( \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \).

So from now on we can assume that our cyclotomic matrix \( A \) has all entries in \( \{-1, 0, 1\} \).
Step 2: \{-1, 0, 1\}-matrices with 0's on the diagonal:

Signed Graphs!

Signed graph = graph where edges have sign +1 or -1.

Example:

Signed graph

\[
\begin{pmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\]

its adjacency matrix
Maximal Connected Cyclotomic Signed graphs.

Theorem There are:
- $S_{14}$
- $S_{16}$
- $T_{2k}$ for all $k \geq 3$. 
Figure 3. The 14-vertex sporadic maximal cyclotomic signed graph $S_4$.

See also Section 12.2.
Figure 4. The hypercube sporadic maximal cyclotomic signed graph size.
Figure 1: The family $T_{2k}$ of 2k-vertex maximal cyclotomic toral tessellations, for $k \geq 3$. (The two copies of vertices A and B should be identified.)
Figure 2. A typical toral tessellation $T_{2k}$; the signed graph $T_{12}$. 
Maximal Connected Cyclotomic Chained Signed Graphs

\[ A \]
\[ 0, \pm 1 \text{ on diagonal} \]

Theorem: There are:

- \( S_7 \)
- \( S_8 \)
- \( S'_8 \)
- \( C_{2k^+} \) for \( k \geq 2 \)
- \( C_{2k^-} \)
FIGURE 7. The three sporadic maximal cyclotomic charged signed graphs $S_7$, $S_8$, $S'$. 

Cyclotomic Matrices
Figure 6. The families of $2k$-vertex maximal cyclotomic cyclindrical tesselations $C_{2k}$ and $C_{2k}$ for $k \geq 2$.

Cyclotomic Matrices
$R_{A_n}$ for graphs

$R_{A_n} \leq 2$ cyclotomic: done combining

$2 < R_{A_n} < \sqrt{2 + 5} = 2.058\ldots$

Graphs $T_a, b, c$ & $H_a, b, c$

$H_a, b, c$

$R_{A_n} > \sqrt{2 + 5}$: dense on $(\sqrt{2 + 5}, \infty)$. 
2 < R_A < 2.019

\[ \overline{T_{a,b,c}} = \]

Figure 1: The tree \( T_{a,b,c} \).
Table 1: The noncyclotomic connected charged signed graphs whose eigenvalues are at most 2.019 in modulus.

<table>
<thead>
<tr>
<th>#</th>
<th>Maximum modulus of eigenvalues</th>
<th>Charged signed graph having corresponding spectral radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.00659...</td>
<td>$T_{1,2,6}$</td>
</tr>
<tr>
<td>2</td>
<td>2.00960...</td>
<td>$10b$</td>
</tr>
<tr>
<td>3</td>
<td>2.01075...</td>
<td>$T_{1,2,7}$</td>
</tr>
<tr>
<td>4</td>
<td>2.01348...</td>
<td>$T_{1,2,8,10a}$</td>
</tr>
<tr>
<td>5</td>
<td>2.01531...</td>
<td>$T_{1,2,10,10c,11a,11b,T_{1,2,9}}$</td>
</tr>
<tr>
<td>6</td>
<td>2.01657...</td>
<td>$10d,T_{1,2,10}$</td>
</tr>
<tr>
<td>7</td>
<td>2.01746...</td>
<td>$T_{1,2,11}$</td>
</tr>
<tr>
<td>8</td>
<td>2.01809...</td>
<td>$T_{1,2,12}$</td>
</tr>
<tr>
<td>9</td>
<td>2.01854...</td>
<td>$T_{1,2,13}$</td>
</tr>
<tr>
<td>10</td>
<td>2.01887...</td>
<td>$T_{1,2,14,12a}$</td>
</tr>
</tbody>
</table>

Theorem: All maximal indecomposable integral symmetric matrices $A$ with $2 < R_A < 2.019$ are equivalent to one of
Salem Number \( \tau \):
“nearest thing to a root of unity”: all but two conjugates on \(|z|=1\).

Conjugates = roots of minimal polynomial.
Here = \(\{\tau, \tau^{-1}, \tau_3, \ldots, \tau_d\}\)

\(\tau > 1, \quad |12 \tau|^2 = \ldots = |12 \tau d|^2 = 1\)

Example \(\tau = 1.176\ldots\), root of
\[3^{10} + 3^9 - 3^7 - 3^6 - 3^5 - 3^4 - 3^3 + 3 + 1\]
\[= (3 - \tau)(3 - \tau^{-1}) \prod_{3=3}^{10} (3 - \tau_3)\]
“Salem Polynomial”
Some graphs give Salem numbers:

\[ 5 \times 9 (\sqrt{3} + i \sqrt{3}) = 3^5 + 3^4 - 3^3 - 3^2 - 3 - 3^0 = 3 + 3 - 3 - 3 - 3 - 3 + 1 \]