Glauber Dynamics for Ising Model on the Complete Graph

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Let $G_n = (V_n, E_n)$ be a graph with $n = |V_n| < \infty$ vertices.

The nearest-neighbor Ising model on $G_n$ is the probability distribution on $\{-1, 1\}^{V_n}$ given by

$$\mu(\sigma) = Z(\beta)^{-1} \exp \left( \beta \sum_{(u,v) \in E_n} \sigma(u)\sigma(v) \right),$$

where $\sigma \in \{-1, 1\}^{V_n}$.

The interaction strength $\beta$ is a parameter which has physical interpretation as $1/T$, where $T = \text{temperature}$. 
Three regimes

High temperature ($\beta < \beta_c$):
low temperature ($\beta > \beta_c$),
Three regimes

critical temperature ($\beta = \beta_c$),
The (single-site) *Glauber dynamics* for $\mu$ is a Markov chain $(X_t)$ having $\mu$ as its stationary distribution.

Transitions are made from state $\sigma$ as follows:

1. a vertex $v$ is chosen uniformly at random from $V_n$.
2. The new state $\sigma'$ agrees with $\sigma$ everywhere except possibly at $v$, where $\sigma'(v) = 1$ with probability

$$
\frac{e^{\beta S(\sigma, v)}}{e^{\beta S(\sigma, v)} + e^{-\beta S(\sigma, v)}}
$$

where

$$
S(\sigma, v) := \sum_{w : w \sim v} \sigma(w).
$$

Note the probability above equals the $\mu$-conditional probability of a positive spin at $v$, given that all spins agree with $\sigma$ at vertices different from $v$.

Note that the Glauber dynamics is reversible with respect to the Gibbs measure $\mu$. 
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Mixing time

For a Markov chain \((X_t)\) on state-space \(\Omega\) and with stationary distribution \(\pi\), write

\[
d(t) = \max_{x \in \Omega} \| \mathbb{P}\{X_t \in \cdot \mid X_0 = x\} - \pi \|_{TV}
\]

Define

\[
t_{\text{mix}}(\epsilon) := \min\{t \geq 0 : d(t) \leq \epsilon\}.
\]

Consider now a sequence of Markov chains, \((X^n_t)\), and write \(d_n(t)\) and \(t_{\text{mix}}^n(\epsilon)\).
A sequence of Markov chains has a cutoff if

\[
\frac{t_{\text{mix}}^n(\epsilon)}{t_{\text{mix}}^n(1 - \epsilon)} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\]

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The Glauber dynamics is said to exhibit a cut-off at \( \{ t_n \} \) with window \( \{ w_n \} \) if \( w_n = o(t_n) \) and

\[
\lim_{\gamma \to \infty} \liminf_{n \to \infty} d_n(t_n - \gamma w_n) = 1,
\]

\[
\lim_{\gamma \to \infty} \limsup_{n \to \infty} d_n(t_n + \gamma w_n) = 0.
\]
For the Glauber dynamics on graph sequences with bounded degree,
\[ t_{\text{mix}}^n = \Omega(n \log n). \]
(T. Hayes and A. Sinclair)

**Conjecture (due to Y. Peres):** If the Glauber dynamics for a sequence of transitive graphs satisfies \( t_{\text{mix}}^n = O(n \log n) \), then there is a cut-off.
Mean field case

Take $G_n = K_n$, the complete graph on the $n$ vertices: $V_n = \{1, \ldots, n\}$, and $E_n$ contains all $\binom{n}{2}$ possible edges.

The total interaction strength should be $O(1)$, so replace $\beta$ by $\beta/n$. The probability of updating to a +1 is then

$$\frac{e^{\beta(S-\sigma(v))/n}}{e^{\beta(S-\sigma(v))/n} + e^{-\beta(S-\sigma(v))/n}}$$

where $S$ is the total magnetization

$$S = \sum_{i=1}^{n} \sigma(i).$$

The statistic $S$ is almost sufficient for determining the updating probability.

Now, in this case $S_t = S(X_t)$ is a Markov chain in its own right; $(S_t)$ will be key to analysis of the dynamics.
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Mean field has $t_{\text{mix}} = O(n \log n)$

A consequence (see, e.g., Aizenman and Holley (1987)) of the Dobrushin-Shlosman uniqueness criterion: For the Glauber dynamics on $K_n$, if $\beta < 1$, then

$$t_{\text{mix}} = O(n \log n).$$

(See also Bubley and Dyer (1998).)
Let \((X^n_t)\) be the Glauber dynamics for the Ising model on \(K_n\). If \(\beta < 1\), then 
\[
    t_{\text{mix}}(\epsilon) = (1 + o(1)) \frac{n \log n}{2(1-\beta)}
\]
and there is a cut-off.

In fact, we show that there is a window of size \(O(n)\) centered about

\[
    t_n = \frac{1}{2(1-\beta)} n \log n.
\]

That is,

\[
    \limsup_{n} d_n(t_n + \gamma n) \to 0 \quad \text{as} \quad \gamma \to \infty.
\]

and

\[
    \liminf_{n} d_n(t_n + \gamma n) \to 1 \quad \text{as} \quad \gamma \to -\infty.
\]
Let \((X^n_t)\) be the Glauber dynamics for the Ising model on \(K_n\). If \(\beta = 1\), then there are constants \(c_1\) and \(c_2\) so that

\[
c_1 n^{3/2} \leq t_{\text{mix}} \leq c_2 n^{3/2}.
\]
If $\beta > 1$, then

$$t_{\text{mix}}^n > c_1 e^{c_2 n}.$$ 

This can be established using Cheeger constant – there is a bottleneck going between states with positive magnetization and states with negative magnetization.

Our results show that once this barrier to mixing is removed, the mixing time is reduced to $n \log n$. 

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Lower bound via Cheeger constant

Let

$$Q(A, A^c) = \sum_{x \in A, y \in A^c} \pi(x)P(x, y).$$

If

$$\Phi_A = \frac{Q(A, A^c)}{\pi(A)},$$

then for $\pi(A) \leq 1/2$,

$$t_{mix} \geq \frac{1}{4\Phi_A}.$$
The bottleneck

Let $A_k$ be those configurations with $k$ spins equal to +1. Then

$$\mu(A_{\lfloor \alpha n \rfloor}) = \frac{1}{Z(\beta)} e^{-n[\phi(\alpha)+o(1)]}.$$  

The function $\phi$ changes shape at $\beta = 1$: 

![Graph showing the function $\phi$ changing shape at $\beta = 1$.]
If $\Omega^+$ are configurations with strictly positive magnetization,

\[
\frac{Q(\Omega^+, (\Omega^+)^c)}{\pi(A_{\lfloor n/2 \rfloor})} \leq \frac{\exp n[\phi(1/2) + o(1)]}{\exp n[\phi(\alpha_0) + o(1)]}.
\]

If $\beta > 1$, there is $\alpha_0$ so that $\phi(\alpha_0) > \phi(1/2)$ and then

\[
\phi_S \leq c_1 e^{-c_2 n}.
\]
If the bottleneck at zero magnetization is removed by truncating the dynamics at zero magnetization, then the chain converges fast:

**Theorem (L.-L.-P.)**

Let $\beta > 1$. Let $(X_t)$ be the Glauber dynamics on $K_n$, restricted to the set of configurations with non-negative magnetization. Then

$$t_{\text{mix}}^n = O(n \log n).$$
Proof idea for low temperature

Use coupling: Show that for arbitrary starting states, can run together two copies of the chain so that the chains meet with high probability in $O(n \log n)$ steps.

- First show that the magnetizations will agree after $O(n \log n)$ steps, when chains are run independently. (Hard part – involves hitting time calculations.)
- After magnetizations agree, couple the chains as below.
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- After magnetizations agree, couple the chains as below.
Write \((X_t)\) and \((\tilde{X}_t)\) for the two chains. We assume that \(S(X_t) = S(\tilde{X}_t)\).

Let \(J\) be the vertex selected for updating in \(X_t\), and let \(s \in \{-1, 1\}\) be the spin used to update \(X_t(J)\).

The \(\tilde{X}\)-chain will also be updated with the spin \(s\) at a site \(\tilde{J}\) which has \(\tilde{X}_t(\tilde{J}) = X_t(J)\), although it will not always be that \(J = \tilde{J}\).

If \(X_t(J) = \tilde{X}_t(J)\), then update both chains at \(J\).

If \(X_t(J) \neq \tilde{X}_t(J)\), then pick \(\tilde{J}\) uniformly at random from

\[
\{i : \tilde{X}_t(i) \neq X_t(i) \text{ and } \tilde{X}_t(i) = X_t(J)\}.
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A coupling (any temperature)

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If $D_t$ is the number of sites where $X_t$ and $\tilde{X}_t$ disagree, then when $D_t \geq 0$, 

$$
\mathbb{E}[D_{t+1} \mid \mathcal{F}_t] \leq \left[1 - \frac{c_1}{n}\right]D_t.
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It takes $O(n \log n)$ steps to drive this expectation down to $\epsilon$. 
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It takes $O(n \log n)$ steps to drive this expectation down to $\epsilon$. 
How exactly do we restrict the dynamics? Whenever in a state with non-negative magnetization and a move is proposed to a state $\eta$ with negative magnetization, we move to $-\eta$ instead.

Show there is a coupling $(X_t^+, \tilde{X}_t^+)$ of restricted dynamics started from $\sigma, \tilde{\sigma}$ such that

$$\limsup_{n \to \infty} \mathbb{P}_{\sigma, \tilde{\sigma}}(\tau_{\text{mag}} > cn \log n) \to 0 \quad \text{as} \quad c \to \infty.$$ 

Here $\tau_{\text{mag}}$ is the first time $t$ such that $S_t^+ = \tilde{S}_t^+$.

By monotonicity, it is enough to consider $\sigma = 0$ and $\tilde{\sigma} = 1$. 
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We couple both these, $S_t^B$ and $S_t^T$ with an equilibrium copy, $S_t$, and we try to get them to “cross over”.

For this, we need to consider hitting times:

$$
\tau_1 = \min\{t \geq 0 : S_t^T \leq s^* + c_1 n^{-1/2}\}
$$

$$
\tau_2 = \min\{t \geq 0 : S_t^B \geq s^* + c_2 n^{-1/2}\},
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where $s^*$ is the unique positive solution of $\tanh(\beta s) = s$.

These hitting times can be analysed, as $nS(X_t^+)/2$ is a birth-and-death chain.
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These hitting times can be analysed, as $nS(X_t^+)/2$ is a birth-and-death chain.
Let \((Z_t)\) be a birth-and-death chain on \(\{0, \ldots, N\}\) with transition probabilities

\[
\begin{align*}
  p_k &= \mathbb{P}(Z_{t+1} - Z_t = +1 \mid Z_t = k), \quad k = 0, \ldots, N - 1, \\
  q_k &= \mathbb{P}(Z_{t+1} - Z_t = -1 \mid Z_t = k), \quad k = 1, \ldots, N, \\
  r_k &= \mathbb{P}(Z_{t+1} - Z_t = 0 \mid Z_t = k), \quad k = 0, \ldots, N.
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Let \(\pi = (\pi(k))\) be the stationary measure.

Let \(Z_t^{(\ell)}\) be a restriction of \(Z_t\) to \(\{0, \ldots, \ell\}\). Note that \(\pi^{(\ell)}(k) \sim \pi(k)\) for \(k = 0, \ldots, \ell\).

Then for \(\ell = 0, 1, \ldots, N - 1\),

\[
\frac{1}{\pi^{(\ell)}(\ell)} = 1 + q_\ell \mathbb{E}_{\ell-1}(\tau_{\ell}).
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Then for \(\ell = 0, 1, \ldots, N - 1\),

\[
\frac{1}{\pi^{(\ell)}(\ell)} = 1 + q_{\ell} \mathbb{E}_{\ell-1}(\tau_{\ell}).
\]
Hitting times for magnetization

Let $\ell^* = \lfloor ns^* \rfloor$. Then

\[
\begin{align*}
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \sqrt{n}(1 + O(n^{-1/2})), \quad 1 \leq \ell \leq C\sqrt{n} \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \frac{C_1 n}{\ell}, \quad C\sqrt{n} \leq \ell \leq \ell^*/2 \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq C_2 n\ell^* - \ell, \quad \ell^*/2 \leq \ell \leq \ell^* - C\sqrt{n} \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \sqrt{n}(1 + O(n^{-1/2})), \quad \ell^* - C\sqrt{n} \leq \ell \leq \ell^* + C\sqrt{n}
\end{align*}
\]
If $S_t = \sum_{i=1}^{n} X_t(i)$, then for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} - S_t | \mathcal{F}_t] \approx -\left[ \frac{S_t}{n} - \tanh(\beta S_t/n) \right].$$
When $\beta < 1$, using the inequality $\tanh(x) \leq x$ for $x \geq 0$ shows that for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} \mid \mathcal{F}_t] \leq S_t \left(1 - \frac{1 - \beta}{n}\right)$$

Need $[2(1 - \beta)]^{-1} n \log n$ steps to drive $\mathbb{E}[S_t]$ to $\sqrt{n}$.

Additional $O(n)$ steps needed for magnetization to hit zero. (Compare with simple random walk.)

Can couple two versions of the chain so that the magnetizations agree by the time the magnetization of the top chain hits zero.

Once magnetizations agree, use a two-dimensional process to bound time until full configurations agree. Takes an additional $O(n)$ steps.
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|\mathbb{E}_z[Z_t] - \mathbb{E}_{\tilde{z}}[Z_t]| \leq \rho^t|z - \tilde{z}|.
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Then \(\nu_t := \sup_{z_0} \text{var}_{z_0}(Z_t)\) satisfies

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This lemma enables us to show that \(\text{var}(S_t) = O(n)\) for \(\beta < 1\) and \(\text{var}(S_t) = O(t)\) for \(\beta = 1\).
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Conjectures:

1. For $\beta < \beta_c$, there is a cut-off.
2. For $\beta = \beta_c$, the mixing time is polynomial in $n$.
   Stronger: $t_{\text{mix}} = O(|V_n|^{3/2})$ for $d > d_c$.
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Levin, Luczak, Peres
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