Graphs classes with given 3-connected components

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Number of planar graphs with $n$ vertices

$$ct \cdot n^{-7/2} \gamma^n n! \quad \gamma \approx 27.23$$

Number of series-parallel graphs

$$ct \cdot n^{-5/2} \gamma^n n! \quad \gamma \approx 9.07$$
Counting graphs
Properties of random graphs
Counting graphs

Properties of random graphs (not Erdös-Renyi model)
Counting graphs

Properties of random graphs (not Erdös-Renyi model)

How many planar graphs?

How many edges in a random planar graph?

Size of largest $k$-connected component

Degree distribution
Counting graphs

Properties of random graphs (not Erdös-Renyi model)

- How many planar graphs?
- How many edges in a random planar graph?
- Size of largest $k$-connected component
- Degree distribution

Graphs are labelled $V = \{1, 2, \ldots, n\}$
Random $\equiv$ uniform distribution
Counting graphs

Properties of random graphs (not Erdős-Renyi model)

How many planar graphs?

How many edges in a random planar graph?

Size of largest $k$-connected component

Degree distribution

Graphs are labelled $V = \{1, 2, \ldots, n\}$

Random $\equiv$ uniform distribution

Efficient algorithm for generating large (10,000 vertices) uniform planar graphs (Fusy)
Number of planar graphs (Giménez, N 2005)

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Number of trees

\[ n^{n-2} \sim ct \cdot n^{-5/2} e^n n! \]
- Number of planar graphs (Giménez, N 2005)

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- Number of trees

\[ n^{n-2} \sim ct \cdot n^{-5/2} e^n n! \]

- Number of outerplanar or series-parallel graphs (Bodirsky, Giménez, Kang, N 2005)

\[ ct \cdot n^{-5/2} \gamma^n n! \quad \gamma_{out} \approx 7.32 \quad \gamma_{sp} \approx 9.07 \]
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-5/2 Tree-like
-7/2 Map-like

Different structural properties
A general framework

\( \mathcal{T} \) family of 3-connected graphs
\( \mathcal{G} \) graphs whose 3-connected components are in \( \mathcal{T} \)
A general framework

\( \mathcal{T} \) family of 3-connected graphs
\( \mathcal{G} \) graphs whose 3-connected components are in \( \mathcal{T} \)

1. \( \mathcal{T} = \text{planar 3-connected} \Rightarrow \mathcal{G} = \text{planar} \)
2. \( \mathcal{T} = \text{planar 3-connected} + K_5 \Rightarrow \mathcal{G} = \text{no } K_{3,3}\text{-minor} \)
3. \( \mathcal{T} = \emptyset \Rightarrow \mathcal{G} = \text{series-parallel} \)
4. \( \mathcal{T} = K_4 \Rightarrow \mathcal{G} = \text{no } W_4\text{-minor} \)
A general framework

\( \mathcal{T} \) family of 3-connected graphs
\( \mathcal{G} \) graphs whose 3-connected components are in \( \mathcal{T} \)

1. \( \mathcal{T} = \) planar 3-connected \( \Rightarrow \) \( \mathcal{G} = \) planar
2. \( \mathcal{T} = \) planar 3-connected \( + K_5 \) \( \Rightarrow \) \( \mathcal{G} = \) no \( K_{3,3} \)-minor
3. \( \mathcal{T} = \emptyset \) \( \Rightarrow \) \( \mathcal{G} = \) series-parallel
4. \( \mathcal{T} = K_4 \) \( \Rightarrow \) \( \mathcal{G} = \) no \( W_4 \)-minor
5. \( \mathcal{T} = \) planar triangulations
\[ T(x, z) = \sum_{n,k} t_{n,k} z^n \frac{x^n}{n!} \quad n \text{ vertices and } k \text{ edges} \]
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\[ B(x, y), \ C(x, y), \ G(x, y), \ D(x, y) S(x, y) \]

\( B \) 2-connected
\( C \) connected
\( G \) graphs
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\[ B(x, y), \ C(x, y), \ G(x, y), \ D(x, y), \ S(x, y) \]

\begin{itemize}
  \item B 2-connected
  \item C connected
  \item G graphs
  \item D networks
  \item S series networks
\end{itemize}
$1 + D(x, y) = (1 + y)e^{S(x,y)} + \frac{2}{x^2} T_z(x,D(x,y))$
Equations

\[ 1 + D(x, y) = (1 + y)e^{S(x,y)} + \frac{2}{x^2} T_z(x, D(x,y)) \]

\[ S(x, y) = (D(x, y) - S(x, y)) x D(x, y) \]
$1 + D(x, y) = (1 + y)e^{S(x, y) + \frac{2}{x^2} T_z(x, D(x, y))}$

$S(x, y) = (D(x, y) - S(x, y)) \times D(x, y)$

$\log \left( \frac{1 + D(x, y)}{1 + y} \right) = \frac{xD(x, y)^2}{1 + xD(x, y)} + \frac{2}{x^2} T_z(x, D(x, y))$
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$$(1 + y)2B_y(x, y) = x^2(D(x, y) + 1)$$
Equations

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\[ xC_x(x, y) = x \exp \left( B_x(xC_x(x, y), y) \right) \]

\[ G(x, y) = \exp(C(x, y)) \]
For fixed $x$, dominant singularity of $T(x, z)$ at $z = r(x)$

$\alpha = \text{singular type}$

$$T(x, z) \sim f(x) \left(1 - \frac{z}{r(x)}\right)^{\alpha}$$

$\alpha = 1/2, \ 3/2 \text{ main singular types}$
For fixed $x$, dominant singularity of $T(x, z)$ at $z = r(x)$

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$$T(x, z) \sim f(x) \left(1 - \frac{z}{r(x)}\right)^{\alpha}$$

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For fixed $y_0$, let $x_0$ dominant singularity of $D(x, y_0)$

$D_0 = D(x_0)$
For fixed $x$, dominant singularity of $T(x, z)$ at $z = r(x)$ is

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For fixed $y_0$, let $x_0$ dominant singularity of $D(x, y_0)$

$D_0 = D(x_0)$

**Theorem**

(1) If $T_z(x, y)$ analytic or singular type $\alpha < 1$ at $(x_0, D_0)$

$$b_n \sim n^{-5/2} R^{-n} n!, \quad c_n \sim n^{-5/2} \rho^{-n} n!, \quad g_n \sim n^{-5/2} \rho^{-n} n!$$
For fixed $x$, dominant singularity of $T(x, z)$ at $z = r(x)$

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**Theorem**

1. *If* $T_z(x, y)$ *analytic or singular type* $\alpha < 1$ *at* $(x_0, D_0)$

   $b_n \sim n^{-5/2} R^{-n} n!, \quad c_n \sim n^{-5/2} \rho^{-n} n!, \quad g_n \sim n^{-5/2} \rho^{-n} n!$

2. *If* $T_z(x, y)$ *singular type* $\alpha = 3/2$ *at* $(x_0, D_0)$, *then either*
   
   (2.1) $b_n \sim n^{-7/2} R^{-n} n!, \quad c_n \sim n^{-7/2} \rho^{-n} n!, \quad g_n \sim n^{-7/2} \rho^{-n} n!$
   
   (2.2) $b_n \sim n^{-7/2} R^{-n} n!, \quad c_n \sim n^{-5/2} \rho^{-n} n!, \quad g_n \sim n^{-5/2} \rho^{-n} n!$
   
   (2.3) $b_n \sim n^{-5/2} R^{-n} n!, \quad c_n \sim n^{-5/2} \rho^{-n} n!, \quad g_n \sim n^{-5/2} \rho^{-n} n!$
For fixed $x$, dominant singularity of $T(x, z)$ at $z = r(x)$
$\alpha = \text{singular type}$

$$
T(x, z) \sim f(x) \left(1 - \frac{z}{r(x)}\right)^{\alpha}
$$

$\alpha = 1/2, \ 3/2 \text{ main singular types}$

For fixed $y_0$, let $x_0$ dominant singularity of $D(x, y_0)$
$D_0 = D(x_0)$

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$$

$R, \rho$ computable positive constants and $\rho < R$
A sample of graph classes

<table>
<thead>
<tr>
<th></th>
<th>$\gamma$</th>
<th>Exponent</th>
<th>Ref.</th>
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</thead>
<tbody>
<tr>
<td>Forests $= \text{Ex}(K_3)$</td>
<td>$e \approx 2.72$</td>
<td>$-5/2$</td>
<td>Standard</td>
</tr>
<tr>
<td>Outerplanar $= \text{Ex}(K_4, K_{2,3})$</td>
<td>7.320</td>
<td>$-5/2$</td>
<td>BGKN</td>
</tr>
<tr>
<td>Series parallel $= \text{Ex}(K_4)$</td>
<td>9.07</td>
<td>$-5/2$</td>
<td>BKGN</td>
</tr>
<tr>
<td>Ex($W_4$)</td>
<td>10.24</td>
<td>$-5/2$</td>
<td></td>
</tr>
<tr>
<td>Ex($K_5 - e$)</td>
<td>12.96</td>
<td>$-5/2$</td>
<td></td>
</tr>
<tr>
<td>Ex($K_2 \times K_3$)</td>
<td>14.13</td>
<td>$-5/2$</td>
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<tr>
<td>Planar</td>
<td>27.226</td>
<td>$-7/2$</td>
<td>GN</td>
</tr>
<tr>
<td>Embeddable in a fixed surface</td>
<td>27.226</td>
<td>??</td>
<td>McDiarmid</td>
</tr>
<tr>
<td>Ex($K_{3,3}$)</td>
<td>27.2293</td>
<td>$-7/2$</td>
<td>GGNW</td>
</tr>
<tr>
<td>Ex($K_{3,3}^+$)</td>
<td>27.2295</td>
<td>$-7/2$</td>
<td>GGNW</td>
</tr>
<tr>
<td>Ex($K_5$)</td>
<td>??</td>
<td>??</td>
<td></td>
</tr>
</tbody>
</table>

$\gamma = \text{Giménez, N}$
BGKN = Bodirsky, Giménez, N, Kung
GGNW = Gerke, Giménez, N, Weissl
Problem: How to estimate $g_{n,\lfloor \mu n \rfloor}$

Fix $y > 0$ suitably depending on $\mu$

$$G(x, y) = \sum g_{n,k} y^k x^n \frac{n^k}{n!}$$

Graphs get weight $y^\#\text{edges}$

Only graphs with $\mu n + O(\sqrt{n})$ get non-negligible weight
Critical phenomena

Planar graphs: for all $1 < \mu < 3$

$$g_{n, \lfloor \mu n \rfloor} \sim \frac{c(\mu)}{\sqrt{n}} n^{-7/2} \gamma(\mu)^n n!$$

SP graphs: always $-5/2$ for $1 < \mu < 2$
Critical phenomena

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SP graphs: always $-5/2$ for $1 < \mu < 2$

$\mathcal{I} = \text{planar triangulations}$

$\mathcal{G} = \text{graphs whose 3-connected members are in } \mathcal{I}$

For $1 < \mu < \mu_0$ exponent is $-5/2$ \quad $\mu_0 \approx 1.87$

For $\mu_0 < \mu < 3$ exponent is $-7/2$
Critical phenomena

Planar graphs: for all $1 < \mu < 3$

$$g_{n, \lfloor \mu n \rfloor} \sim \frac{c(\mu)}{\sqrt{n}} n^{-7/2} \gamma(\mu)^n n!$$

SP graphs: always $-5/2$ for $1 < \mu < 2$

$\mathcal{T} =$ planar triangulations
$\mathcal{G} =$ graphs whose 3-connected members are in $\mathcal{T}$

For $1 < \mu < \mu_0$ exponent is $-5/2$ \hspace{1cm} $\mu_0 \approx 1.87$

For $\mu_0 < \mu < 3$ exponent is $-7/2$

Dominant singularity comes from $T(x, z)$ or
from a branch point when solving

$$\log \left( \frac{1 + D(x, y)}{1 + y} \right) = \frac{xD(x, y)^2}{1 + xD(x, y)} + \frac{2}{x^2} T_z(x, D(x, y))$$
Random graphs

For all classes under study

1. Number of edges is asympt. normal with linear mean and variance
2. Number of components is asympt. Poisson
3. Largest conn. component is $n - \mathcal{O}(1)$

Difference is in size of largest 2- and 3-connected components
Banderier, Flajolet, Schaeffer, Soria (2001)
Random maps, . . . and Airy phenomena
Banderier, Flajolet, Schaeffer, Soria (2001)
Random maps, ... and Airy phenomena

Largest 2-connected component in random maps follows asympt. Airy distribution
Related to Airy function \( y'' - xy = 0 \)
Banderier, Flajolet, Schaeffer, Soria (2001)
Random maps, ... and Airy phenomena

Largest 2-connected component in random maps follows asympt. Airy distribution
Related to Airy function $y'' - xy = 0$

Critical composition schemes in maps

$$M(z) = C(uH(z)), \quad H(z) = z(1 + M(z))^2$$

$u$ marks size of 2-connected core
We can apply the techniques from Banderier et al. to planar graphs and

Size of largest 2-connected component has mean $\sim \alpha_0 n$

Fluctuations are of order $O(n^{2/3})$
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Size of largest 2-connected component has mean $\sim \alpha_0 n$
Fluctuations are of order $O(n^{2/3})$

For series-parallel graphs size of largest 2-connected component is $O(1)$ whp