Counting with Gaussian Integrals and cluster expansions

David C. Brydges

April 8, 2008, Newton Institute

Abstract

This is a review, beginning with examples of how to express various combinatorial problems, including self-avoiding walks and loops in graphs, in terms of Gaussian integrals. The second half will be a review of techniques such as tree graph formulas for approximating the resulting integrals.

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□

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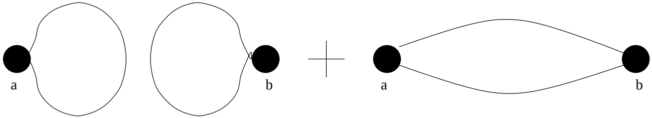

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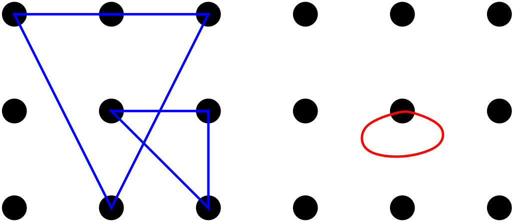

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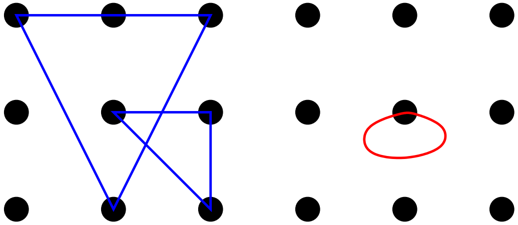

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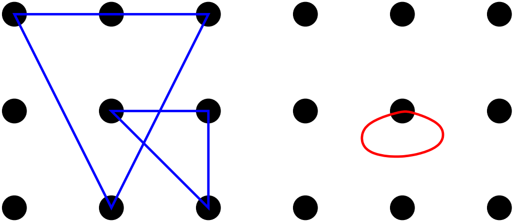

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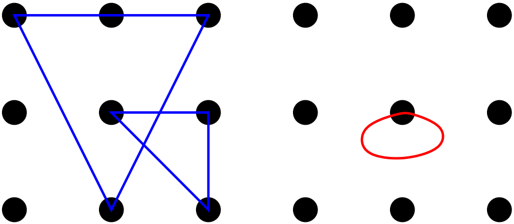

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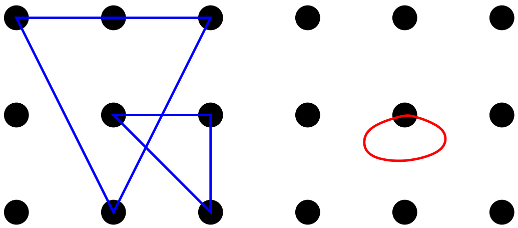

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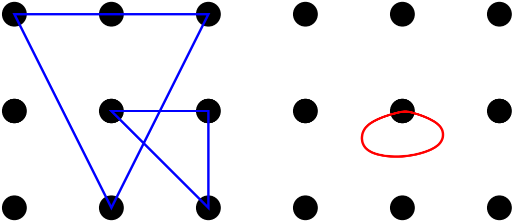

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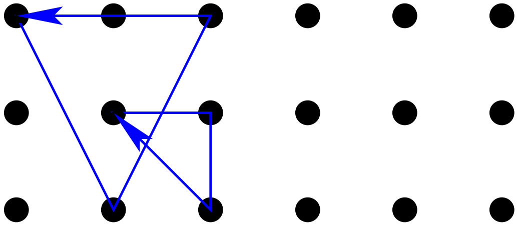

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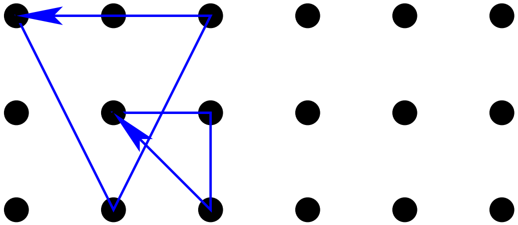

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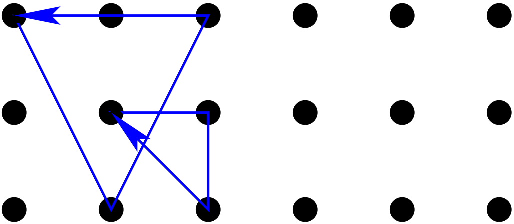

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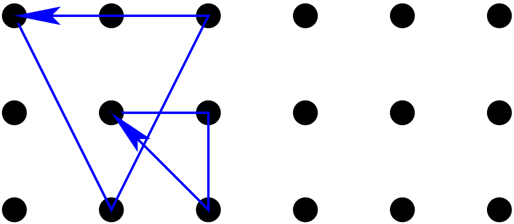

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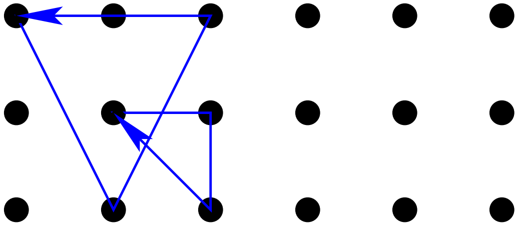

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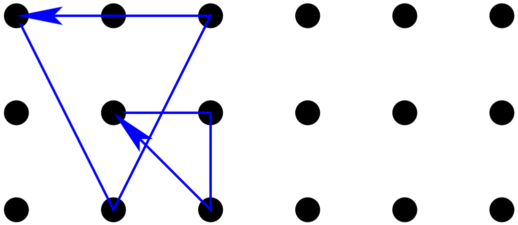

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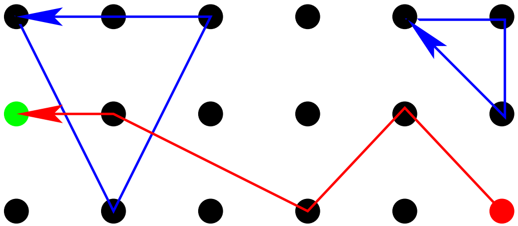

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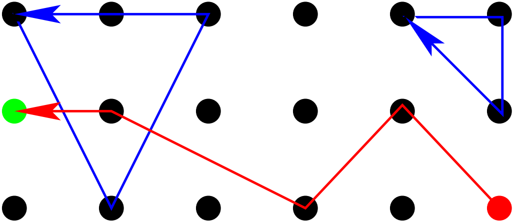

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Self-Avoiding Walk, Cohomology and Cosmology!!

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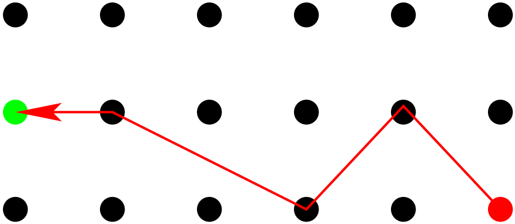

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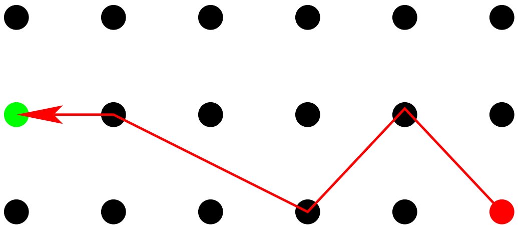

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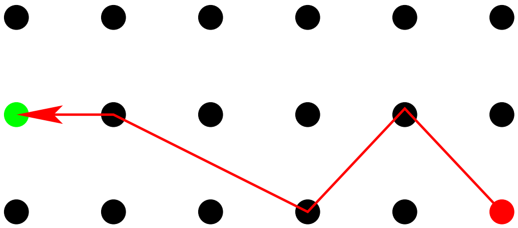

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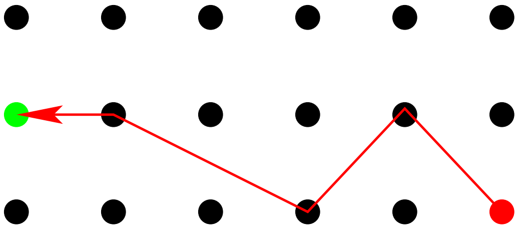

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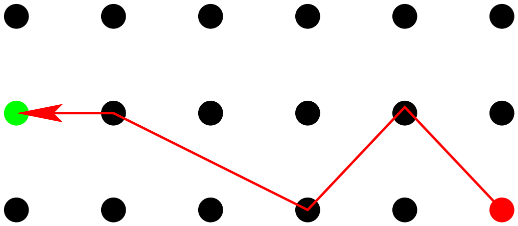

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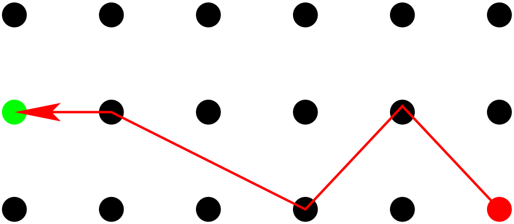

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Supersymmetry $Q = \iota_V + d$ implies zero vacuum energy.

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weight of Steiner tree, vertices $x_1, \ldots, x_n$

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Mayer expansion then implies pressure is analytic in I.

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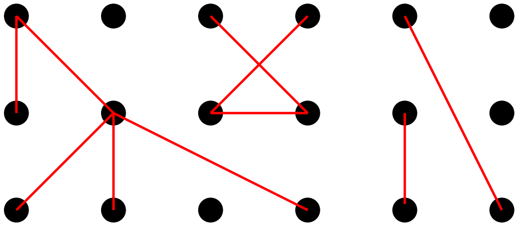

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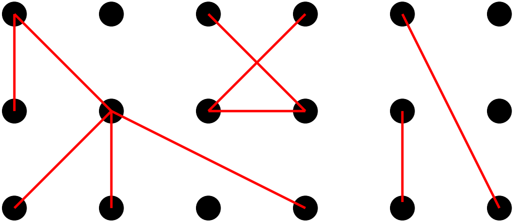

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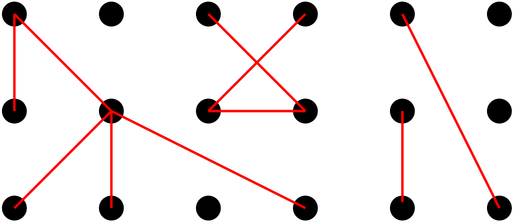

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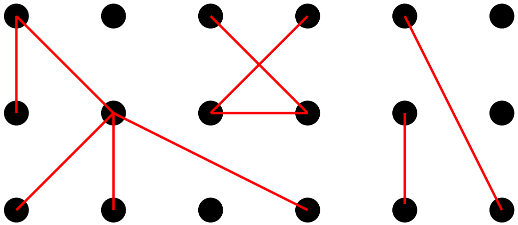

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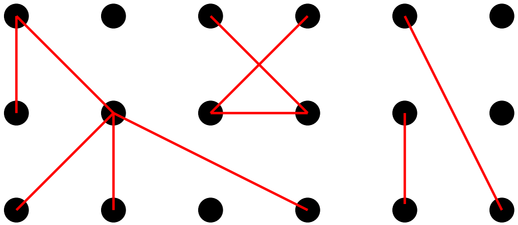

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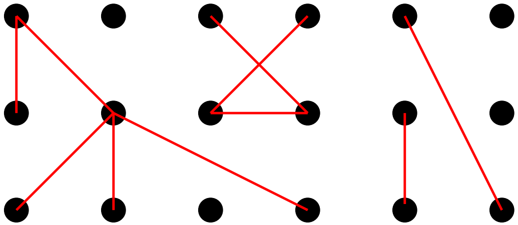

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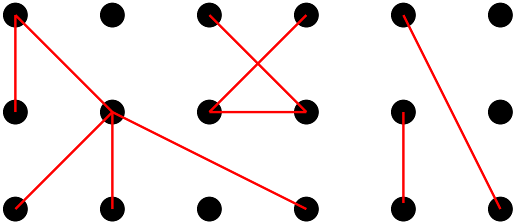

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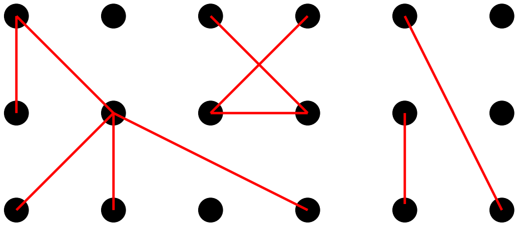

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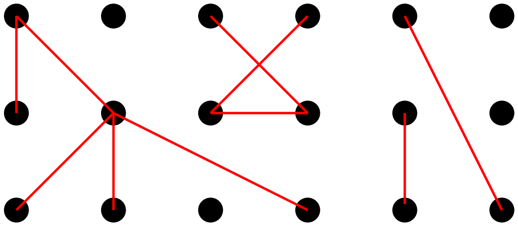

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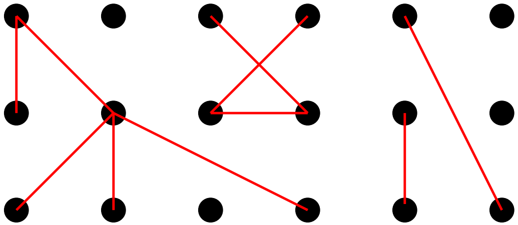

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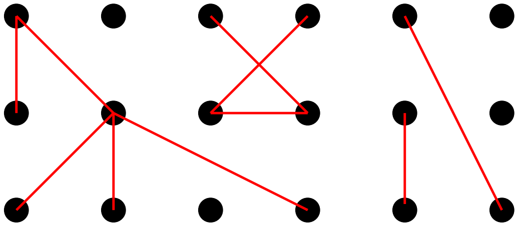

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Under the law of $\mathbb{E}_{F,s}$, I_x in different trees are independent.

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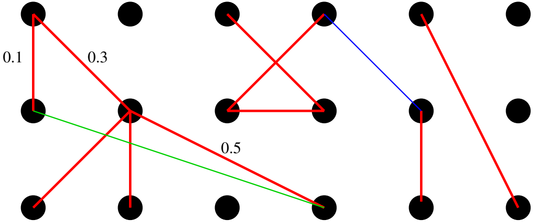

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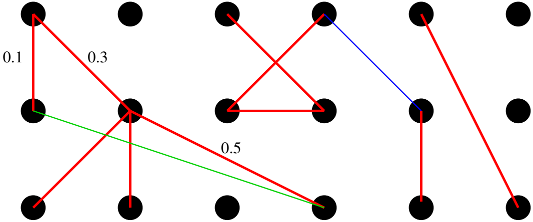

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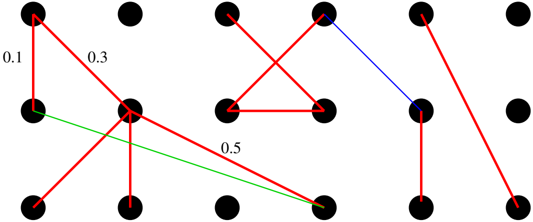

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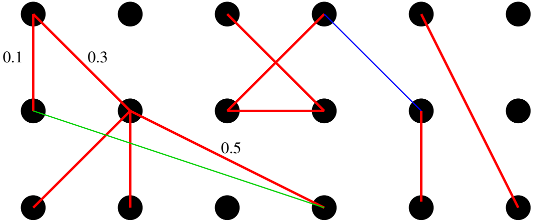

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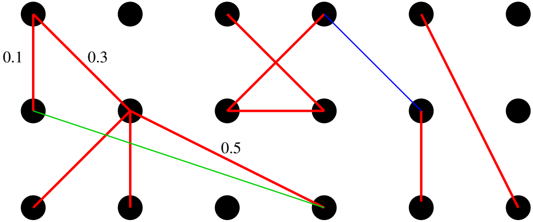

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$$\sigma_F(s)_{xy} = \begin{cases} \inf\{s_e : e \in \text{ path in F joining } x \text{ and } y\} \\ 0 \text{ if no path} \end{cases} \quad (1)$$

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In particular, there exists Gaussian expectation $\mathbb{E}_{F,s}$ such that $e^{\sum \sigma_{F,xy}(s)\Delta_{xy}} = \mathbb{E}_{F,s}$.

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