Mayer expansion of repulsive polymer models

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$-1 \leq F(\Gamma, \Gamma') \leq 0$

PLAN OF THE LECTURE:

① Mayer expansion of

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1, \dots, \Gamma_N)} \prod_{i=1}^{N} w_{\Gamma_i} \sum_{\substack{G \text{ graph} \\ \text{on } (1,2,\dots,N)}} F_G$$

Th. $Z = \exp\left(\sum_{\mathcal{C}} F_{\mathcal{C}} w_{\mathcal{C}}\right)$

$\mathcal{C} = (\Gamma_1, \dots, \Gamma_M) \quad M \geq 1 \quad w_{\mathcal{C}} = \frac{1}{M!} \prod_{i=1}^{M} w_{\Gamma_i}$

$$F_{\mathcal{C}} = \sum_{\substack{G \text{ connected} \\ \text{on } (1,\dots,M)}} F_G$$

$$F_G = \prod_{(i,j) \in G} F(\Gamma_i, \Gamma_j)$$

② K.P. formula for

$a_\Gamma = \sum_{\mathcal{C}} F_{\Gamma,\mathcal{C}} w_{\mathcal{C}} \quad F_{\Gamma,\mathcal{C}} = \prod_{\Gamma_i \in \mathcal{C}} \left(1 + F(\Gamma, \Gamma_i)\right) - 1$

Th. $a_\Gamma = \sum_{\Gamma'} F(\Gamma, \Gamma') w_{\Gamma'} \int_0^1 e^{a^t_{\Gamma'}} dt$

3 Resummation of $\sum F_\varphi w_\varphi$

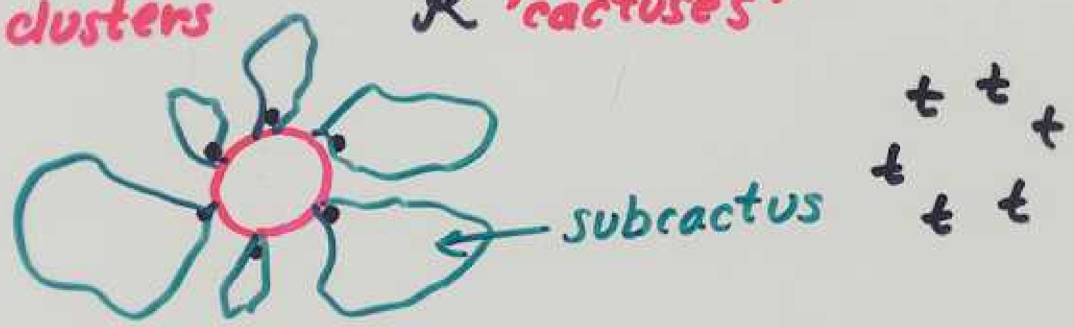
soft repulsion case

$$F_\varphi = \sum_{G \text{ connected}} F_G = \sum_{\mathcal{T} \text{ left tree on } (1,\dots,M)} F_{\mathcal{T}}$$

4 Resummation

hard repulsion by intersection

$$\sum_{\varphi \text{ clusters}} F_\varphi w_\varphi = \sum_{\mathcal{K} \text{ 'cactuses'}} F_{\mathcal{K}} w_{\mathcal{K}}$$ [all $\mathcal{K}$ the same sign]



NOTATION

$$Z = \sum_{\{\Gamma_i\}} \prod_i w_{\Gamma_i}$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod_{i=1}^{N} w_{\Gamma_i} \prod_{i<j} \left(1 + F(\Gamma_i, \Gamma_j)\right)$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod w_{\Gamma_i} \sum_{G \text{ graph on } (1,\dots,N)} F_G$$

$$\varphi \equiv (\Gamma_1,\dots,\Gamma_N) \qquad w_\varphi \equiv \frac{1}{N!} \prod_{i=1}^{N} w_{\Gamma_i}$$

$$F_G = \prod_{(i,j)\in G} F(\Gamma_i, \Gamma_j)$$

Using $$e^{x_1+\dots+x_k} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(y_1,\dots,y_N)} \prod_{j=1}^{N} y_j$$

$y_j \in \{x_1,\dots,x_k\}$

we get

THEOREM (MAYER)

$$Z = \exp\Big(\sum_{\varphi} F_\varphi w_\varphi \Big) \qquad F_\varphi = \sum_G F_G$$

$\varphi = (\Gamma_1,\dots,\Gamma_M)$ $M \geq 1$ ← connected

Example: $$1+x = \exp\Big(\sum_{M=1}^{\infty} \frac{1}{M!} \sum_{G \text{ connected}} (-1)^{|G|} x^M \Big)$$

KOTECKY–PREISS–CAMMAROTA–SEILER–DOBRUSHIN–...

Put $$a_\Gamma = \sum_{\text{all } \varphi} F_{\Gamma,\varphi} (F_\varphi w_\varphi)$$

who is 'responsible' for touching?

$$F_{\Gamma,\varphi} = \prod_i \big(1+F(\Gamma,\Gamma_i)\big) - 1$$

$= -1$ in the hard repulsion case

Standard trick: auxiliary parameter

$$0 \leq t \leq 1$$

DEF: Consider the "(t,Γ) relaxed" model: replace $F(\Gamma,\Gamma')$ by softer $tF(\Gamma,\Gamma')$

Remind $0 \leq 1 + tF(\Gamma,\Gamma') \leq 1$

$t=0$ INDEPENDENCY OF Γ $\quad$ $t=1$ ORIGINAL MODEL

Consider also the "$[t,\Gamma]$ relaxed" model where all $w_{\Gamma'}$ are replaced by smaller quantities $(1+tF(\Gamma,\Gamma'))w_{\Gamma'}$ (including $\Gamma'=\Gamma$)

THEOREM

$$\frac{\partial}{\partial t}a_\Gamma^{(t,\Gamma)} = \sum_{\Gamma'} F(\Gamma,\Gamma')\,w_{\Gamma'}\,e^{a_{\Gamma'}^{[t,\Gamma]}}$$

NOTE. If

$$b_\Gamma > \sum |w_{\Gamma'}|\,|F(\Gamma,\Gamma')|\,e^{b_{\Gamma'}}$$

then iterative method of solution converges

RESUMMATION - SOFT REPULSION

'Left trees' instead of Penrose trees

DEFINITION.

Left tree on an ordered set $(1,2,..,m) \equiv M$ is defined recursively by decomposing M into a singleton $\{1\}$ (the 'chief' of the tree) and a collection of disjoint subtrees

$$L(m) = \sum_{\text{all decompositions } \{M_i\} \text{ of } M\setminus\{1\}} \prod L(m_i)$$

! SAME FOR CYCLES:

$$C(m) = \sum \prod C(m_i)$$

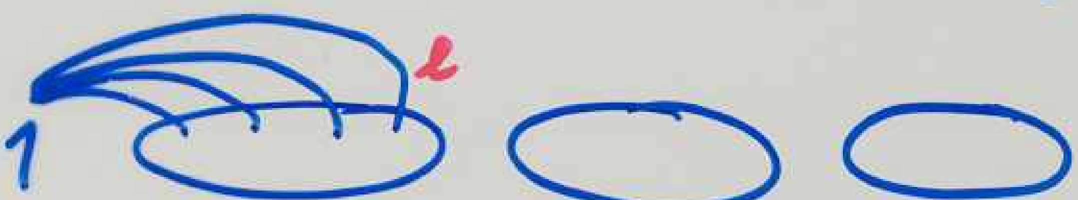
HENCE the number of left trees on M, $|M| = m$ is $(m-1)!$

and

$$\log(1-x) = -\sum_{m=1}^{\infty} \frac{(m-1)!}{m!} x^m$$

Put $F_{1,\mathcal{T}_j} = \prod_{\Gamma_k \in \mathcal{T}_j} \left(1 + F(\Gamma_1, \Gamma_k)\right) - 1$

for a left tree $\mathcal{T} = \{1 \& \{\mathcal{T}_j\}\}$



$$(1-1)^{\ell} - 1 = -1$$

THEOREM

$$\sum_{G \text{ connected on } (1,2,\dots,m)} F_G = \sum_{\mathcal{T}} F_{\mathcal{T}}$$

where for any left tree we define

$$F(\mathcal{T}) = \prod_{\substack{u \text{ subchief of } \mathcal{T} \\ U \text{ subordinates to } u}} F(u, U)$$

$$F_{u,U} = \prod_{v \in U} \left(1 + F(u, v)\right) - 1$$

Hence

$$Z = \exp\Big(\sum_{\mathcal{C}=(\Gamma_1,\dots,\Gamma_m)} F_{\mathcal{C}}\, w_{\mathcal{C}} \Big)$$

$$F_{\mathcal{C}} = \sum_{\mathcal{T} \text{ left trees on } (\Gamma_1,\dots,\Gamma_m)} F(\mathcal{T})$$

same sign $(-1)^{m-1}$

HARD REPULSION, CASE

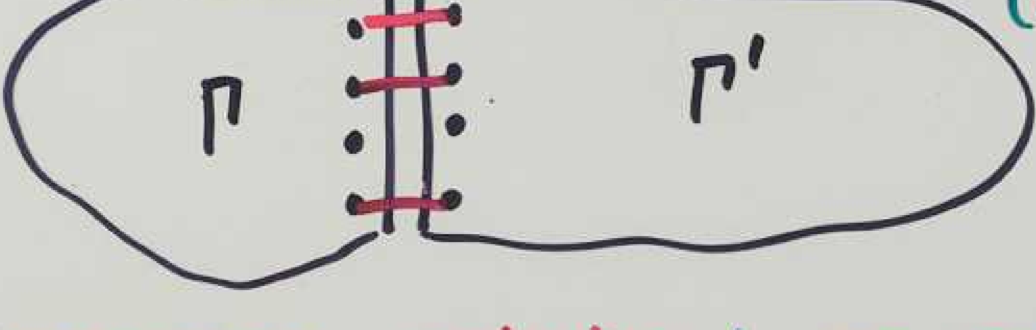
$$F(\Gamma,\Gamma') = \begin{cases} -1 & \text{IFF } \operatorname{supp}\Gamma \cap \operatorname{supp}\Gamma' \neq \emptyset \\ 0 & \phantom{\text{IFF }} \operatorname{supp}\Gamma \cap \operatorname{supp}\Gamma' = \emptyset \end{cases}$$

Forget left trees.

SUBSTITUTION OF SPECIES:

QUILTED CLUSTERS, bonds of G replaced by BIPARTITE GRAPHS

$(1-1)^4 - 1 = -1$



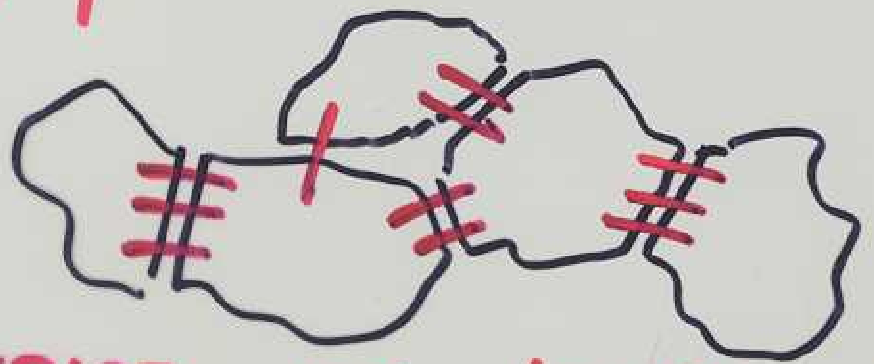
AUXILIARY ordering $\preccurlyeq$ OF THE 'LATTICE' Λ WHERE POLYMERS LIVE

$\underline{t}$ first point of Λ

PUT BRACKETS INTO THE SUM

$$\sum_{Q} F_Q w_Q$$

of quilted clusters $\mathcal{C}$



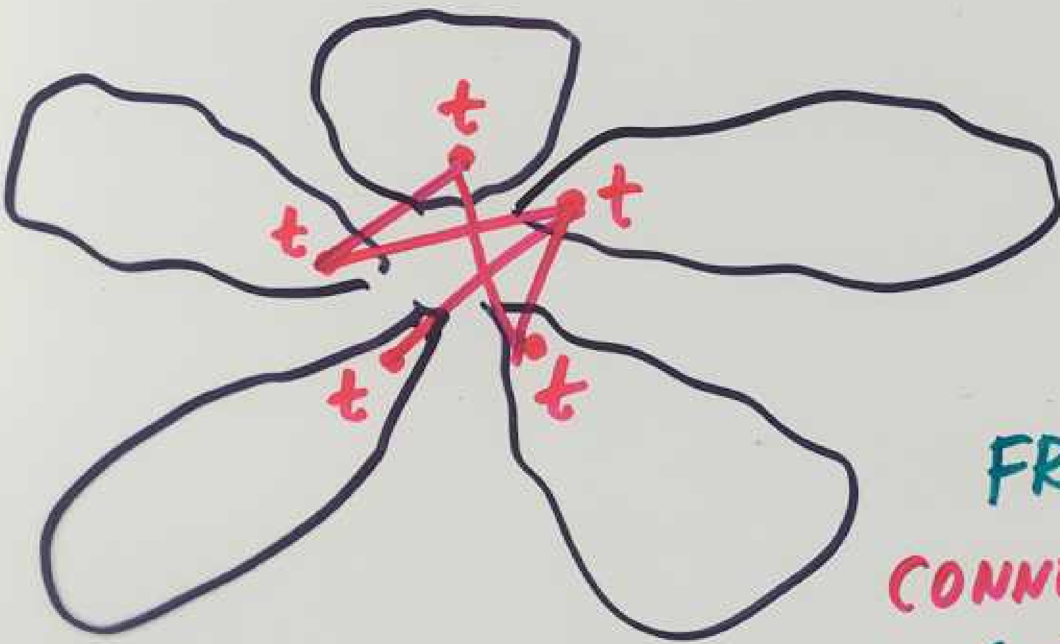
ERASE red bonds $t \leftrightarrow t$

$$\sum_{\{Q_i\}} \prod_i \left(\sum F_{Q_i}\right) (-1)^{\text{No of bonds } t \leftrightarrow t}$$

TERMS corresponding to components like can be erased

bond $t \leftrightarrow t$
here is optional
does not influence
the bracketing

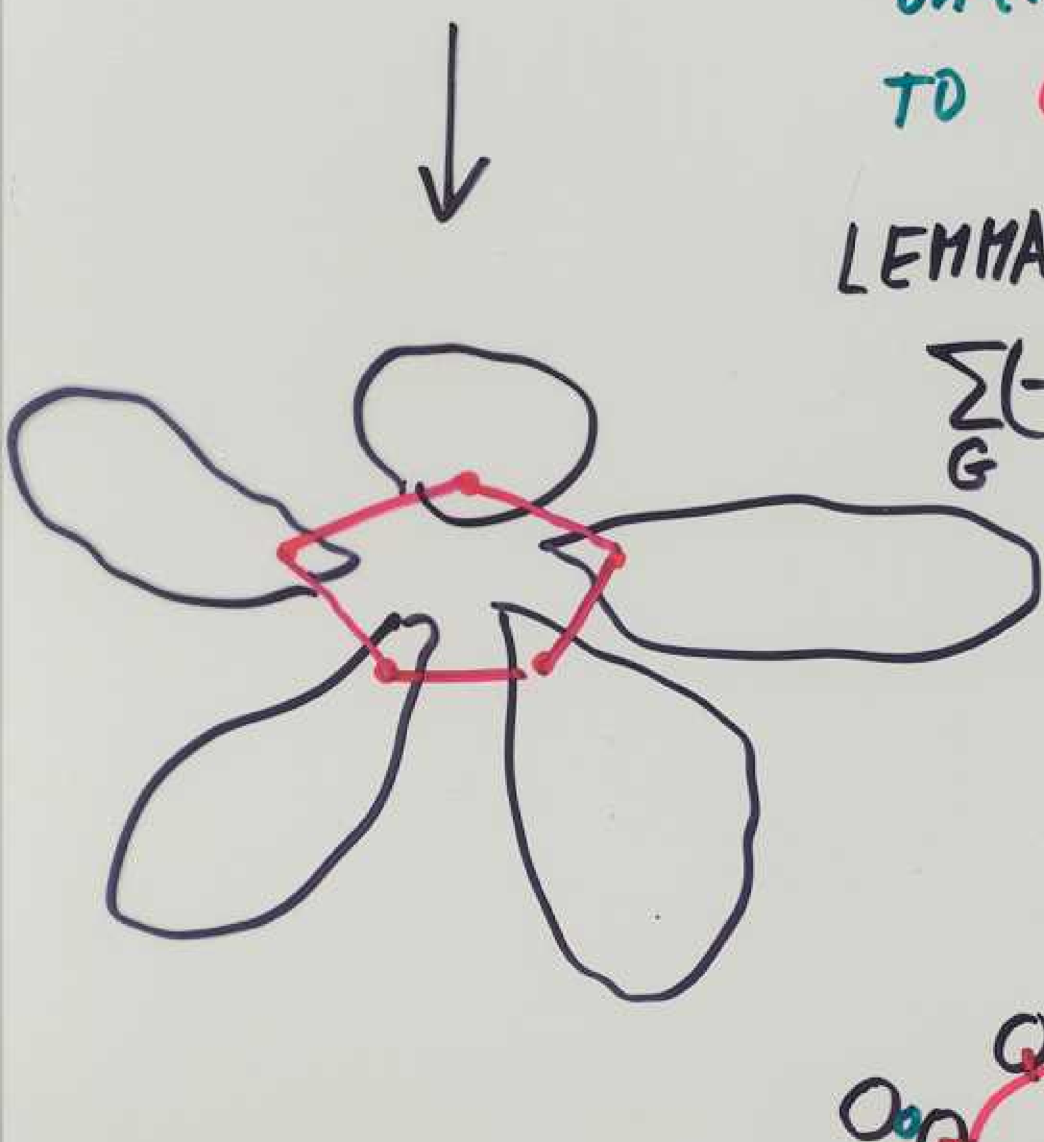
What remains are quilted clusters of the following typ

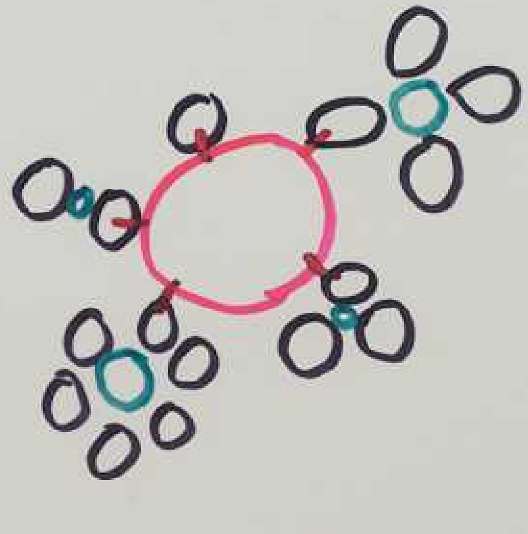


FROM
CONNECTED GRAPHS
on $(1,2,\dots,k)$
TO CYCLES

LEMMA

$$\sum_G (-1)^{|G|} = (-1)^{k-1}(k-1)!$$





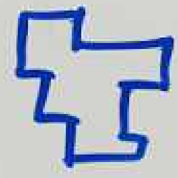
E.T.C.

This method is close to exact solutions

$$\det(\mathbb{1}-W) = \exp\left(-\sum \frac{1}{n} \operatorname{Tr} W^n\right)$$ (9)

(closed) walks interpreted as cactuses of cycles

ONSAGER $\quad Z = \sum \prod_i w_{P_i}$



$\ulcorner \ \sqrt{i}$

$\urcorner \ \sqrt{-i}$

$\text{I} \ 1$

cactuses of contours interpreted as walks in dual lattice