An Extensor Tree Theorem and a Tutte Identity for Graphs with Distinguished Port Edges

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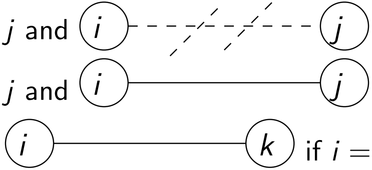
Matrix Tree Theorem

Laplacian (Kirchhoff) matrix of a graph

$$L_{ij} = \begin{cases} 0 & \text{if } i \neq j \text{ and } \\ -1 & \text{if } i \neq j \text{ and } \\ \sum 1 & \text{all } k \qquad \text{if } i = j \end{cases}$$



Theorem

Each cofactor ($1 \leq a, b \leq \#$ vertices) counts spanning trees, i.e.,

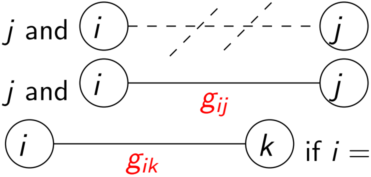
$$(-1)^{(a+b)} \det L(\overline{a}, \overline{b}) = \sum_{T:\text{spanning trees}} 1$$

(Many proofs and applications dating to Kirchhoff and Maxwell ...)

Matrix Tree Theorem

Laplacian (Kirchhoff) matrix of a graph with weights

$$L_{ij} = \begin{cases} 0 & \text{if } i \neq j \text{ and} \\ -1 g_{ij} & \text{if } i \neq j \text{ and} \\ \sum 1 g_{ik} & \text{all } k \end{cases} \quad \text{if } i = j$$



Theorem

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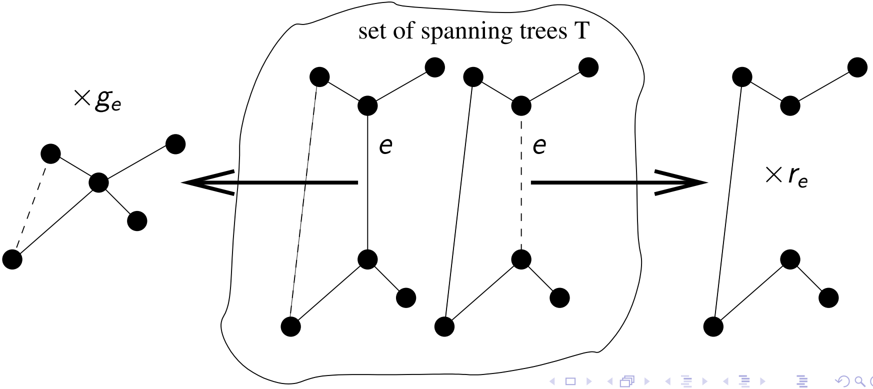
$$(-1)^{(a+b)} \det L(\overline{a}, \overline{b}) = \sum_{T\text{:spanning trees}} 1 \prod_{e \in T} g_e$$

(Many proofs and applications dating to Kirchhoff and Maxwell ...)

Weighted Tutte Functions: Example and Additive Identity

$$M(\text{(graph) } G) = \sum_{\substack{T\text{ : spanning trees}\\ \text{in } G}} \prod_{e \in T} g_e \prod_{e \in \overline{T} = E \setminus e} r_e = \sum_T g_T r_{\overline{T}}$$

For edge e: $M(G) = g_e M(G/e(\text{contraction})) + r_e M(G \setminus e\ (\text{deletion}))$



Tutte Functions satisfy 2 Identities

(Additive (del/contr)) and Multiplicative: $M(G_1 \oplus G_2) = M(G_1)M(G_2)$

- Some Tutte functions: Chromatic polynomial, Pott's model partition functions, many others.
- Popular theory for graphs (graphic matroids), matroids.
- The range is usually a commutative ring.
- Tree counting has applications to physics, but are there physical motivations for the matrix tree theorem?
- We present a Tutte function into an (anticommutative) exterior algebra. (i.e., algebra with anticommutative Grassmann-Berezin variables) It generalizes det $L(\bar{a}, \bar{b})$. (I know of no other interesting non-ring examples...)

Our Tutte function's VALUE (on an electrical network graph) represents the solution to a classical physics problem.

Maxwell's Rule (simplest case)

R_{ab} = Equivalent electrical resistance between a and b.
We make p denote a "dummy" or added edge we will call a port to demark pair a, b. We will use $R_{p,p}$ instead of R_{ab}.
$R_{p,p}$ is NOT a Tutte function, but....

$R_{p,p} = M(G/p) : M(G \setminus p)$ when resistance of each e is $r_e : g_e$.

- $M(G/p)$ enumerates spanning trees including p.
- $M(G \setminus p)$ enumerates spanning trees excluding p.
- If G is not connected, "spanning trees" would be "graphic matroid bases," i.e., full rank trees.
- (Ratio notation ":" is used because this is valid when either $M(G/p)$ or $M(G \setminus p)$ is zero.)

(Port voltage and current observed in lab)

$$R_{p,p} = M(G/p) : M(G \backslash p) \equiv \begin{bmatrix} M(G \setminus p) & -M(G/p) \end{bmatrix} \begin{bmatrix} v_p \\ i_p \end{bmatrix} = 0$$

- The solution space, projected on the v_p, i_p coordinates, is the orthogonal complement of the (1-dim) row space of matrix $[M(G \setminus p) \quad - M(G/p)]$.
- Let's present the row space as the 1-form $M(G \setminus p)\mathbf{p}_v^* \quad - M(G/p)\mathbf{p}_\iota^*$, also denoted $M(G \setminus p)dv_p - M(G/p)di_p$.

Why Bother with Exterior Algebra?

$M(G/p)$ and $-M(G \setminus p)$ each satisfy the Tutte Equations (with $e \neq p$) separately, so OUR 1-FORM satisfies:

$$M_E(G) = g_e M_{E\setminus e}(G/e) + r_e M_{E\setminus e}(G \setminus e) \ \ (p \notin E)$$

Result

This generalizes to any number of ports.
When there are p ports the objects are $p-$forms over $\mathbb{R}[r,g]^{2p}$
Each of the $\binom{2p}{p}$ coefficients satisfies its own Matrix Tree Theorem.
Each coefficient, and the $p-$form, is a function of all graphs with distinguished "port" edges labelled with the common set P.

The coefficients are components $m_{ijk\ldots}$ of an antisymmetric tensor of rank p in a $2p$ dim. space.
(We will drop the distinction between k-forms and k-vectors; we work in the exterior algebra over KS)

Applications: Case of 2 Port Edges

$$\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{aligned} M_E = & (m_{1,1}\mathbf{i}_1^* + m_{1,2}\mathbf{i}_2^* + m_{1,3}\mathbf{v}_1^* + m_{1,4}\mathbf{v}_2^*)\wedge \\ & (m_{2,1}\mathbf{i}_1^* + m_{2,2}\mathbf{i}_2^* + m_{2,3}\mathbf{v}_1^* + m_{2,4}\mathbf{v}_2^*) \end{aligned}$$

$$= \begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} i_1^* \wedge i_2^* + \cdots$$

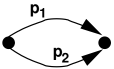
M_E has $\binom{4}{2} = 6$ coefficients, one for each 2×2 minor.

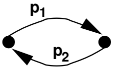
Transfer resistance in terms of minors (= coeffs. of M_E)

$$\begin{bmatrix} \text{Matrix} & . & . \\ . & \text{expr. of} & . \\ . & . & M_E \end{bmatrix} \begin{bmatrix} i_1 = 1 \\ i_2 = 0 \\ v_1 = \text{ don't care} \\ v_2 = -R_{p_2,p_1} \end{bmatrix} = 0$$

$$R_{p_2,p_1} = -\frac{v_2}{i_1} = \frac{M_E[31]}{M_E[34]} = \frac{\sum_{\text{common trees in } G\backslash p_1/p_2 \text{ and } G\backslash p_2/p_1} \pm g_T r_{\overline{T}}}{\sum_{\text{trees in } G\backslash\{p_1,p_2\}} g_T r_{\overline{T}}}$$

The general Maxwell's rule includes the sign rule:

$-$ if G/T looks like 

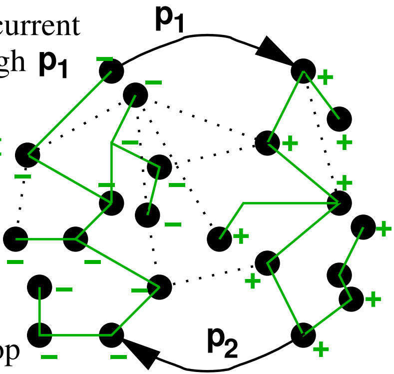
$+$ if G/T looks like 

The sign rule is intuitive

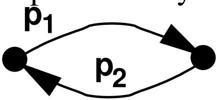
When unit current flows through **p₁**

this forest contributes a positive amount

to the voltage drop along **p₂**



When **this forest** is contracted and the other non–port edges deleted, the resulting oriented matroid minor is represented by:



Application: Rayleigh Identity

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_e}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \,\&\, G/f} - T^{-}_{G/e \,\&\, G/f} \right)^2$$

$T^{\pm}_{G/e \,\&\, G/f}$ enumerate the $\pm$ common spanning trees. Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G}$$

ETC...
(Applications???)

Result

For all graphs $G(E,P)$ with distinguished edge subset P, $G(E,P) \to M_E(G)$ is an extensor-valued function that obeys the 2 Tutte Equations (with sign corrections expressed combinatorially) over exterior algebra, where the multiplication is anticommutative.

Plan

1. Deploy exterior algebra to realize linear (graphic) oriented matroids, minors (deletion/contraction) and dualization.
2. Use Kirchhoff's and Ohm's laws to define $\mathbf{M}_E$ for a graph.
3. Analyze (2) in terms of (1). The generically non-zero terms are characterized by graphic matroid properties of relevent resistor edge and port sets. The signs are characterized by oriented matroid properties.
4. (Definition of $\mathbf{M}_E(\mathbf{N})$ and our result apply to any extensor with ground set $P \cup E$, but the coefficients of $\pm g_F r_{\overline{F}}$ might not be 1.)

Extensors, Linear Subspaces and Matroids

Exterior Algebra

The exterior algebra over an $|S|$ dimensional linear space KS can be generated by $|S|$ independent, anticommuting rank 1 basis vectors S (over K).
Multiplication is multilinear and for $s_1, s_2 \in S$,
$s_1 \wedge s_2 = s_1 s_2 = -s_2 s_1$.

Extensor

A rank k (fully) decomposible element is the exterior product of k linearly independent vectors, i.e., non-zero elements of KS.

Key fact

The $k-$ dimensional linear subspaces of KS correspond one-to-one with classes of rank-k (non-zero) extensors equivalent under non-zero scalar (K) multiplication.

Extensors and Subspaces of KS

$$r\text{-dim row subspaces in } KS \text{ of full row rank } N = \overbrace{\begin{bmatrix} .. & .. & .. \\ .. & N_{ie} & .. \\ .. & .. & .. \end{bmatrix}}^{S}$$

correspond 1-1 to the extensors equiv., under non-zero scalar multiplication, to the extensor:

$$\mathbf{N} = (N_{1,s_1}\mathbf{s}_1 + \cdots N_{1,s_{|S|}}\mathbf{s}_{|S|}) \wedge (N_{2,s_1}\mathbf{s}_1 + \cdots N_{2,s_{|S|}}\mathbf{s}_{|S|}) \wedge \cdots \wedge (N_{r,s_1}\mathbf{s}_1 + \cdots N_{r,s_{|S|}}\mathbf{s}_{|S|})$$

The subspace corresponding to extensor **N**

$$(x_{s_1}, \cdots, x_{s_{|S|}}) \in \text{ row space } (N) \text{ iff}$$

$$\mathbf{N} \wedge (x_{s_1}\mathbf{s}_1 + \cdots + x_{s_{|S|}}\mathbf{s}_{|S|}) = 0$$

Subspaces, (Oriented) Matroids and Extensors

The (oriented) matroids represented by the (signed) column dependencies of matrices N and N' are the SAME if N and N' have the same row spaces.
There are a dozen or so "cryptomorphic" ways to present the combinatorial data of a(n) (oriented) matroid.
We choose (unimodular) N to represent linearly over K the graphic matroid, so S names the graph edges, whose ...

- Circuits = Minimal lin. dep. sets of columns = (directed) "circles" in the graph;
- Bases = Max. independent sets of columns = Max. rank spanning forests = Spanning trees if the graph is connected;
- We take a full row rank N, so a sequence of columns is a $(\pm)$ basis if the corresponding minor in non-zero (with $\pm$ sign).

N's (Oriented) Matroid Bases in Extensor Terms

When we multiply out extensor $\mathbf{N}$ written in terms of basis S of KS, and collect common monomials using $\mathbf{b}_1\mathbf{b}_2... = \epsilon(\sigma)\mathbf{b}_{\sigma_1}\mathbf{b}_{\sigma_2}..$ we can express

$$\mathbf{N} = \sum_{B\subset S} N[B]\mathbf{b}_1\mathbf{b}_2... = \sum_{B\subset S} N[B]\mathbf{B}$$

$N[B]$ is a minor of matrix N. Each N[B]**B** is independent of the order chosen for $B \subset S$. ($N[B] = N_B$ in tensor component notation.)

$N[B] \neq 0$ iff B is a basis. ($\chi(B) = \text{sign}(N[B]) \in \{+, -, 0\}$ is the chirotope of an oriented matroid.)

In fact, one oriented matroid "cryptomorphism" is a sign $\chi(B)$ for each $r-$sequence B which is alternating and which satisfies the signed basis exchange combinatorial condition implied by the Grassmann-Plucker identity:

$$[a_1 a_2 \cdots a_r][b_1 b_2 \cdots b_r] = \sum_{i=1}^{r} [b_i a_2 \cdots a_r][b_1 \cdots \hat{b}_i a_i \cdots b_r]$$

Deletion

Plan

- Our Tutte-like equations are algebraic.
- Given $e \in S$ and an extensor N realizing a(n) (oriented) matroid, define deletion and contraction so the result is a well-defined extensor and can be used in algebraic expressions. Same for dualization.

Deletion is easy.
But in a matroid, $\backslash e$ reduces the rank when e is an isthmus (coloop).
We define $\mathbf{N} \setminus e = \mathbf{0}$ (the zero extensor) if the rank is reduced.

Contraction

In matrix terms ...

Row-reduce to eliminate e as a column. Geometrically, intersect the row space with a hyperplane.

Write $\mathbf{N} = \mathbf{N}_1 \wedge e + \cdots$. Then $\mathbf{N}/e = \mathbf{N}_1$. (This is Berezin's $\partial/\partial e$ up to sign.)

Contraction of e reduces the rank by 1 except when e is a (self-)loop (zero column in the matrix). In that case, $\mathbf{N}/e = \mathbf{0}$.
NB. Zero-rank matroids (all loops) have $\mathbf{N} = \mathbf{1}$ (multiplicative identity).

Dualization

Linear Motivation of Duality

When an (oriented) matroid is presented by the row subspace L within KS of a matrix with columns labelled by S, its dual matroid is presented by the orthogonal complementary subspace $L^\perp$.

Bases in (oriented) matroids

- A rank-k matroid can be specified by which subsets $B \subseteq S$ with $|B| = k$ are (and are not) bases $\mathcal{B}$ (i.e., maximal independent).
- An oriented matroid $\mathcal{N}$ can be specified by which ordered k-sequences B from S are $(\mathcal{N}[B] = 0)$ not independent, $(\mathcal{N}[B] = +)$ positive, and $(\mathcal{N}[B] = -)$. The chirotope function is antisymmetric and satisfies a signed basis-exchange axiom iff it defines an oriented matroid.

Ways to define (Oriented) Matroid Duals

Duals

- Matroid: $\mathcal{B}^* = \{S \setminus B | B \in \mathcal{B}\}$
- Oriented Matroid: $\mathcal{N}^*[\overline{B}] = \pm\epsilon(\overline{B}B)\mathcal{N}[B]$ for $(|S| - k) -$ sequences $\overline{B}$. B is an arbitrary sequence complementary to $\overline{B}$; order of B doesn't matter.
- But $\mathcal{N}^*[]$ and $-\mathcal{N}^*[]$ define $\mathcal{N}^*$ equally well.

But we want * on extensors to be well-defined and satisfy $(N/e)^* = (N^* \setminus e)$.

Ground Set Orientations

We (arbitrarilly) declare with ϵ_U which parity class of permutations of each subset of U is positive: $\epsilon_U(a_1 a_2 \ldots) = \epsilon(\sigma)\epsilon_U(a_{\sigma_1} a_{\sigma_2} \ldots)$ for all permutations σ of all finite subsets $A = \{a_1, a_2, \ldots\}$.

Motivation:

An orientation of a manifold is a consistant specification of which ordered tangent space bases are called positive or "right handed coordinate systems".
So, pseudo-forms such as volume can be defined in a way that the the volume of a sequence of vectors is positive when the sequence is a "right handed coodinate system."

We use a ground set orientation ϵ to define extensor dual so the oriented matroid relationships between deletion, contraction and dualization translate into identities on extensor operations.

Definition of Extensor Dual, Matroid-like Identities

Given $\mathbf{N}(S)$,

$$\mathbf{N}^{\perp}[X] = \mathbf{N}^{\perp_\epsilon}[X] = \mathbf{N}[S']\epsilon(S'\ X),$$

where S' is any permutation of the elements in $S \setminus X$.

Some resulting identities have sign corrections!

$$(\mathbf{N} \setminus X)^{\perp} = \epsilon(S')\epsilon(S'X)\ \ (\mathbf{N}^{\perp}/X)$$

$$(\mathbf{N}/X)^{\perp} = \epsilon(S')\epsilon(S'X)(-1)^{|X|\ (|S|-\rho\mathbf{N})}\ \ (\mathbf{N}^{\perp} \setminus X)$$

$$(\mathbf{N}_1\mathbf{N}_2)^{\perp} = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{\rho\mathbf{N}_1^{\perp}\rho\mathbf{N}_2}\ \mathbf{N}_1^{\perp}\mathbf{N}_2^{\perp}$$

Defining M_E

$$\upsilon_r(\mathbf{e}) = r_e\mathbf{e} \text{ for } e \in E \text{ and } \upsilon_r(\mathbf{p}) = \mathbf{p}_\upsilon \text{ for } p \in P.$$
$$\iota_g(\mathbf{e}) = g_e\mathbf{e} \text{ for } e \in E \text{ and } \iota_g(\mathbf{p}) = \mathbf{p}_\iota \text{ for } p \in P. \qquad (1)$$

Given a ported extensor $\mathbf{N}(P, E)$, a ground set orientation ϵ and dual operator $\perp_\epsilon$, parameters g_e and r_e for each $e \in E$, and ϵ-preserving functions υ_r and ι_g defined above, let

$$\mathbf{M}(\mathbf{N}) = \iota_g(\mathbf{N})\, \upsilon_r(\mathbf{N}^{\perp_\epsilon}) \text{ and } \mathbf{M}_E(\mathbf{N}) = \mathbf{M}(\mathbf{N})/E$$

Electricity! - When N is graphic.

Variables $x_e, e \in E$ represent values such that $g_e x_e$ is the current through edge e and $r_e x_e$ is the voltage across e. Thus Ohm's law is expressed with resistance $r_e : g_e$.

$\iota_g(\mathbf{N})$ expresses Kirchhoff's current law. $\upsilon_r(\mathbf{N}^{\perp_\epsilon})$ expresses Kirchhoff's voltage law.

Contraction by E expresses eliminating the variables $x_e, e \in E$ leaving p independent linear constraints on the $2p$ variables for the port currents and voltage drops.

The r_e, g_e parametrized extensor-valued function $\mathbf{M}_E(\mathbf{N})(P_\upsilon \uplus P_\iota)$ of ported extensor $\mathbf{N} = \mathbf{N}(P, E)$ has the following properties:

1. Given $\mathbf{N}_1(P_1, E_1)$ and $\mathbf{N}_2(P_2, E_2)$ with $E = E_1 \uplus E_2$ and $P = P_1 \uplus P_2$,

$$\mathbf{M}_E(\mathbf{N}_1 \ \ \mathbf{N}_2)(P, E) = \epsilon(P_1 P_2 E)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\, \mathbf{M}_{E_1}(\mathbf{N}_1) \ \ \mathbf{M}_{E_2}(\mathbf{N}_2).$$

2. If $e \in E$ and $E' = E \setminus e$ then

$$\mathbf{M}_E(\mathbf{N}) = \epsilon(PE)\epsilon(PE') \left(g_e \mathbf{M}_{E'}(\mathbf{N}/e) + r_e \mathbf{M}_{E'}(\mathbf{N} \setminus e) \right).$$

3. Let $E = \emptyset$. The Plücker coordinates of $\mathbf{M}_\emptyset(\mathbf{N})(P_\iota \dot\cup P_\upsilon)$ satisfy

$$\mathbf{M}_\emptyset(\mathbf{N})[I_\iota V_\upsilon] = \mathbf{M}[I_\iota V_\upsilon] = \epsilon(\overline{V}\ \ V)\ \mathbf{N}[I]\mathbf{N}[\overline{V}].$$

   for all $I \subseteq P$ and $V \subseteq P$.
   (NB: Each $\mathbf{N}[A]\mathbf{N}[B] \neq 0$ iff A and B are common bases in the matroid represented by $\mathbf{N}$.
   For graphic and other unimodular oriented matroids, each $\mathbf{N}[] = \pm 1$ or 0.)
4. $\mathbf{M}_E(\mathbf{0}) = \mathbf{0}$.

Corollary

Let $(\mathbf{N}/A|P) = \mathbf{N}/A \setminus (E \setminus A)$ be the extensor obtained by contracting $A \subseteq E$ and deleting the rest of E, leaving an extensor with ground set P.
$M_E(\mathbf{N}/A|P)$ with $E = \emptyset$ is a result of applying the reductions in the additive identity repeatedly until there are no more $e \in E$.

$$\epsilon(PE)\mathbf{M}_E(\mathbf{N}) = \epsilon(P) \sum_{\substack{A \subseteq E \,:\, \rho_{\mathbf{N}} A = |A|, \\ \rho\mathbf{N} - \rho(\mathbf{N}/A|P) - \rho_{\mathbf{N}} A = 0}} \mathbf{M}_\emptyset(\mathbf{N}/A|P) g_A r_{\overline{A}}.$$

The signs cancel in a telescoping product.
When $\mathbf{N}$ is graphic, each non-zero $\mathbf{M}_\emptyset(\mathbf{N}/A|P)$ represents the behavior of an electrical network with ports only!
Intuitively, the behavior of the resistor network is the exterior sum of behaviors of certain networks obtained by contracting a forest F of resistors and deleting $E \setminus F$, weighted by $g_F r_{E \setminus F}$.

1. The generic Matrix Tree Theorem: Given $\mathbf{N} = \mathbf{N}(P, E)$, and sequences $I \subseteq P$, $V \subseteq P$, and $\overline{V} = P \setminus V$,

$$\epsilon(\overline{V}\ V)\epsilon(PE)\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] = \epsilon(P) \sum_{A \subseteq E} \mathbf{N}[IA]\mathbf{N}[\overline{V}A]g_A r_{\overline{A}}.$$

The only non-zero terms in this sum are those for which both $A \mathbin{\dot\cup} I$ and $A \mathbin{\dot\cup} \overline{V}$ are bases in the matroid of $\mathbf{N}$.

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2. $\epsilon(PE)\mathbf{M}_E^\epsilon(\pm\mathbf{N})[P_\iota]$ enumerates the bases of $\mathcal{N}(\mathbf{N}/P)$, assuming P is independent in the matroid $\mathcal{N}(\mathbf{N})$, by

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3. $\mathbf{M}_E^\epsilon(\pm\mathbf{N})[Q]$ is constant under sign change of $\pm\mathbf{N}$, and is alternating in E, ϵ and Q.

4. $\epsilon(PE)\mathbf{M}_E^\epsilon(\pm\mathbf{N})[Q]$ is constant under sign change of $\pm\mathbf{N}$ and under changes or reorderings of ϵ or E; it is alternating in P and in Q.

2 Grassmann variables for each edge

Instead of Smith's "protovoltage" x_e for each edge, we could have used voltage x_e and current $\overline{x_e}$. Ohm's law is $(g_e x_e - r_e \overline{x_e}) = 0$.

$M(G)$ with Ohm's Law Explicit:

$$\text{Current Laws: } \mathbf{N} = \bigwedge_{i=0}^{\rho(G)} \left(\sum_E N_{i,e} \overline{\mathbf{x}}_e + \sum_P N_{i,p} \mathbf{p}_\iota\right)$$

$$\text{Voltage Laws: } \mathbf{N}^\perp = \bigwedge_{i=0}^{\rho^*(G)} \left(\sum_E N^\perp_{i,e} \mathbf{x}_e + \sum_P N^\perp_{i,p} \mathbf{p}_\upsilon\right)$$

$$\mathbf{M}_{\text{Ohm's law explicit}} = \mathbf{N}\mathbf{N}^\perp \bigwedge_{e \in E} (g_e x_e - r_e \overline{x_e})$$

Extracting Tree sums

Let $\phi\overline{\phi} = \bigwedge_{p \in P} \mathbf{p}_\upsilon \mathbf{p}_\iota \bigwedge_{e \in E} \mathbf{x}_e \overline{\mathbf{x}_e}$
For 2 sequences of port names I and V, $|I| + |V| = |P|$ (not necessarily disjoint), the coefficient (Plucker coordinate, tensor component) named by $I_\iota V_\upsilon$ in $\mathbf{M}_E$ is given by the Grassmann-Berezin integral:

$$\pm \int \mathcal{D}(\phi\overline{\phi}) \mathbf{I^C}_\iota \mathbf{V^C}_\upsilon \mathbf{M}_{\text{Ohm's..}}$$

where $I^C = P \setminus I$ and $V^C = P \setminus V$.

- Each of these enumerates, with homogenous $\pm \prod r$ and $\prod g$ weights, certain trees. Each is a full-row minor (determinant) in the matrix of M.
- When I, V partition P, all signs are the same.
- Each satisfies Tutte's deletion/contraction and direct sum identities.

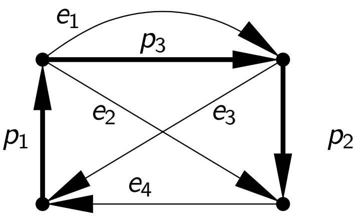
A Grassmann Polynomial that satisfies anticommutative Tutte equations

Berezin Integral Notation

$$\int dx_1 dx_2 \ldots F \text{ denotes } \frac{\partial}{\partial x_1} \circ \frac{\partial}{\partial x_2} \circ \cdots F$$

(When the signs are corrected properly) the integrand obtained by "integrating out" the variables $\mathbf{x}_e$ and $\overline{\mathbf{x}}_e$ satisfies Tutte's equations as a polynomial in Grassmann-Berezin variables.

Example



$$
\begin{array}{c} \\ N = \end{array}
\begin{array}{ccc|cccc}
p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \\
\end{array}
$$

$$
N = \left[\begin{array}{ccc|cccc}
-1 & 0 & +1 & +1 & +1 & 0 & 0 \\
0 & +1 & -1 & -1 & 0 & +1 & 0 \\
-1 & -1 & +1 & +1 & 0 & 0 & +1
\end{array}\right]
$$

$$
\mathbf{N} = \begin{array}{c}
(-\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{e}_1 + \mathbf{e}_2)\cdot \\
(\mathbf{p}_2 - \mathbf{p}_3 - \mathbf{e}_1 + \mathbf{e}_3)\cdot \\
(-\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{e}_1 + \mathbf{e}_4)
\end{array}
$$

Next, we write one totally unimodular matrix $N^{\perp}$ for the canonical dual. We have checked that the sign was chosen properly.

$$N^{\perp} = \begin{array}{c} \\ \end{array} \begin{bmatrix} \overset{p_1}{0} & \overset{p_2}{0} & \overset{p_3}{+1} & \overset{e_1}{-1} & \overset{e_2}{0} & \overset{e_3}{0} & \overset{e_4}{0} \\ +1 & +1 & +1 & 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 & 0 & +1 & 0 \end{bmatrix}$$

We abbreviate labels $p_{\iota 1}$ and $p_{\upsilon 1}$ by i_1 and v_1, etc.

$$M(N) = \begin{array}{c} \begin{array}{cccccccccc} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \left[\begin{array}{ccc|ccc|cccc} -1 & 0 & +1 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & -g_1 & 0 & g_3 & 0 \\ -1 & -1 & +1 & 0 & 0 & 0 & g_1 & 0 & 0 & g_4 \\ \hline 0 & 0 & 0 & 0 & 0 & +1 & -r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & +1 & +1 & 0 & 0 & 0 & r_4 \\ 0 & 0 & 0 & 0 & +1 & +1 & 0 & -r_2 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & +1 & 0 & 0 & r_3 & 0 \end{array}\right] \end{array}$$

We calculate $\mathbf{M}_E(\mathbf{N})$ by doing ring operations on rows to eliminate all but one non-zero entry in each E column in $M(N)$. The result is that

$$g_1 g_2 g_3 g_4 r_1^6 r_2 r_3 r_4 \mathbf{M}(\mathbf{N})$$

is equal to the following extensor in matrix form:

$$
\begin{array}{cccccccccc}
i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4
\end{array}
$$

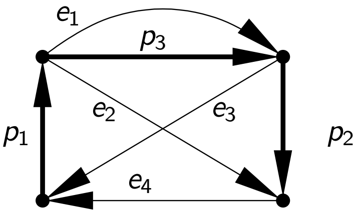
$$
\left[\begin{array}{ccc|ccc|cccc}
-r_1 r_2 & 0 & r_1 r_2 & 0 & g_2 r_1 & g_1 r_2 + g_2 r_1 & 0 & 0 & 0 & 0 \\
0 & r_1 r_3 & -r_1 r_3 & -g_3 r_1 & 0 & -g_1 r_3 - g_3 r_1 & 0 & 0 & 0 & 0 \\
-r_1 r_4 & -r_1 r_4 & r_1 r_4 & -g_4 r_1 & -g_4 r_1 & g_1 r_4 - g_4 r_1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & g_1 & -g_1 r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & g_4 r_1 & g_4 r_1 & g_4 r_1 & 0 & 0 & 0 & g_4 r_1 r_4 \\
0 & 0 & 0 & 0 & g_2 r_1 & g_2 r_1 & 0 & -g_2 r_1 r_2 & 0 & 0 \\
0 & 0 & 0 & g_3 r_1 & 0 & g_3 r_1 & 0 & 0 & g_3 r_1 r_3 & 0
\end{array}\right]
$$

After some cancellation, we can read off the answer from the 3×6 upper left submatrix, which is a matrix presentation of the extensor $r_1^2 \mathbf{M}_E(\mathbf{N})$:

$$\begin{array}{c} \begin{array}{cccccc} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 \end{array} \\ \left[\begin{array}{ccc|ccc} -r_1 r_2 & 0 & r_1 r_2 & 0 & g_2 r_1 & g_1 r_2 + g_2 r_1 \\ 0 & r_1 r_3 & -r_1 r_3 & -g_3 r_1 & 0 & -g_1 r_3 - g_3 r_1 \\ -r_1 r_4 & -r_1 r_4 & r_1 r_4 & -g_4 r_1 & -g_4 r_1 & g_1 r_4 - g_4 r_1 \end{array}\right] \end{array}$$

One can notice that every order 3 minor is a multiple of r_1^2.

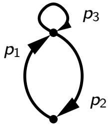
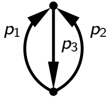
Example graph (again)



Here are examples of Plücker coordinates, which can be calculated from the above matrix as order 3 minors divided by r_1^2.

$$
\begin{aligned}
\mathbf{M}_E(\mathbf{N})[v_1 v_2 v_3] &= g_1 g_2 g_3 r_4 + g_1 g_2 g_4 r_3 + g_1 g_3 g_4 r_2 + g_2 g_3 g_4 r_1 \\
\mathbf{M}_E(\mathbf{N})[i_1 v_2 v_3] &= (g_1 r_3 + g_3 r_1)(g_2 r_4 + g_4 r_2) \\
\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3] &= -g_1 g_4 r_2 r_3 + g_2 g_3 r_1 r_4
\end{aligned}
$$

Observe $\mathbf{M}_E(\mathbf{N})[v_1 v_2 v_3]$ is the basis enumerator for $\mathcal{N}(\mathbf{N}) \setminus P$.

Graph Minor	Extensor Minor	Term in $\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3]$
	$\mathbf{N}/\{e_1, e_4\}\vert P$	$-g_1 g_4 r_2 r_3$
	$\mathbf{N}/\{e_2, e_3\}\vert P$	$+g_2 g_3 r_1 r_4$