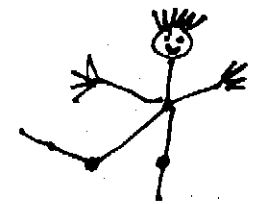
Playing Wick Game



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joint work with Mihyun Kang

't Hooft (74), Brézin, Itzykson, Parisi, Zuber (78), Di Francesco,

Ginsparg, Zinn-Justin, Harer, Zagier, Kontsevich,
Kenyon, Okounkov, ...

Di Francesco : lecture notes Lando, Zvonkin : book

Tutte, Cori, Vauquelin, Schaeffer, Bousquet-Melou ...

Ercolani, McLaughlin, Guionnet, Maurel-Segala ...

Gaussian integral

$$\langle f(x)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} f(x)\,dx \qquad \langle 1\rangle = 1$$

$$\langle x^k\rangle = \frac{\partial^k}{\partial\rho^k}\langle e^{x\rho}\rangle\Big|_{\rho=0} = \frac{\partial^k}{\partial\rho^k} e^{\frac{\rho^2}{2}}\Big|_{\rho=0} = \begin{cases} 0, & k=2n+1 \\ (2n-1)(2n-3)\dots 1 & \end{cases}$$

source integral: $\langle e^{x\rho}\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-\rho)^2}{2}+\frac{\rho^2}{2}}\,dx =$

$$e^{\frac{\rho^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-\rho)^2}{2}}\,dx = e^{\frac{\rho^2}{2}}$$

$M_{ij} = \overline{M_{ji}}$

Gaussian matrix integral. $M = (M_{ij})$ Hermitian

$$dM = \prod_i M_{ii} \prod_{i<j} d\,\mathrm{Re}(M_{ij})\, d\,\mathrm{Im}(M_{ij})$$ standard Haar measure

$$\langle f(M) \rangle = \frac{1}{Z_0(N)} \int e^{-N \mathrm{Tr}\left(\frac{M^2}{2}\right)} f(M)\, dM$$

$$Z_0(N) = \int e^{-N \mathrm{Tr}\left(\frac{M^2}{2}\right)} dM$$

over all $N \times N$ Hermitian matrices

$$\left\langle e^{\mathrm{Tr}(MS)} \right\rangle = \frac{1}{Z_0(N)} \int e^{-N \mathrm{Tr}\left(\frac{1}{2}\left(M - \frac{S}{N}\right)^2\right)} \cdot e^{\frac{\mathrm{Tr}(S^2)}{2N}} dM =$$

$$\boxed{e^{\frac{\mathrm{Tr}(S^2)}{2N}}}$$

source integral

$$\left\langle e^{\mathrm{Tr}(MS)} \right\rangle = e^{\frac{\mathrm{Tr}(S^2)}{2N}} \qquad \left.\frac{\partial}{\partial S_{ji}} e^{\mathrm{Tr}(MS)}\right|_{S=0} = M_{ij}$$

$$\langle M_{ij} M_{kl} \rangle = \left.\frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} e^{\frac{\mathrm{Tr}(S^2)}{2N}}\right|_{S=0} =$$

$$\left.\frac{\partial}{\partial S_{ji}} \left[\left(\frac{\partial}{\partial S_{lk}} \frac{\sum_{m,n} S_{mn} S_{nm}}{2N} \right) \cdot e^{\frac{\mathrm{Tr}(S^2)}{2N}} \right]\right|_{S=0} =$$

$$\left.\frac{\partial}{\partial S_{ji}} \frac{S_{kl}}{N} e^{\frac{\mathrm{Tr}(S^2)}{2N}}\right|_{S=0} = \frac{\Delta_{il} \Delta_{jk}}{N}$$

$\boxed{\text{WICK's theorem}}$ $$\left\langle \prod_{I} M_{ij} \right\rangle = \sum_{P} \prod_{((ij)(kl)) \in P} \frac{\Delta_{il} \Delta_{jk}}{N}$$

P : pairing of elements of I

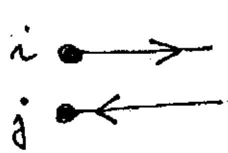
Wick game

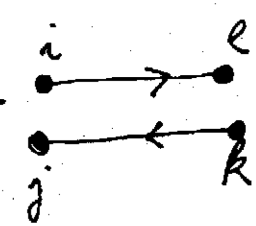
$$f(M) = \sum_{I \subseteq N\times N} a_I \prod_{(i,j)\in I} M_{ij}$$

$$\langle f \rangle = \sum_{I} a_I \sum_{P \text{ pairing of } I} \prod_{(p,q)\in P} \frac{\Delta_{p_1 q_2}\,\Delta_{p_2 q_1}}{N}$$

$p = (p_1\, p_2)$
$q = (q_1\, q_2)$

Example

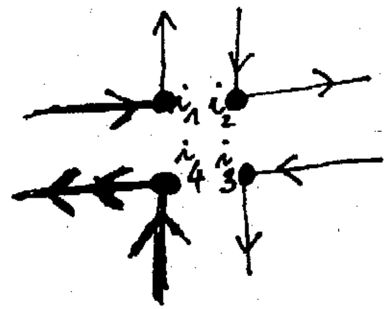
$M_{ij} \approx$ 

$\langle M_{ij}\, M_{kl} \rangle \approx \frac{1}{N}$ 

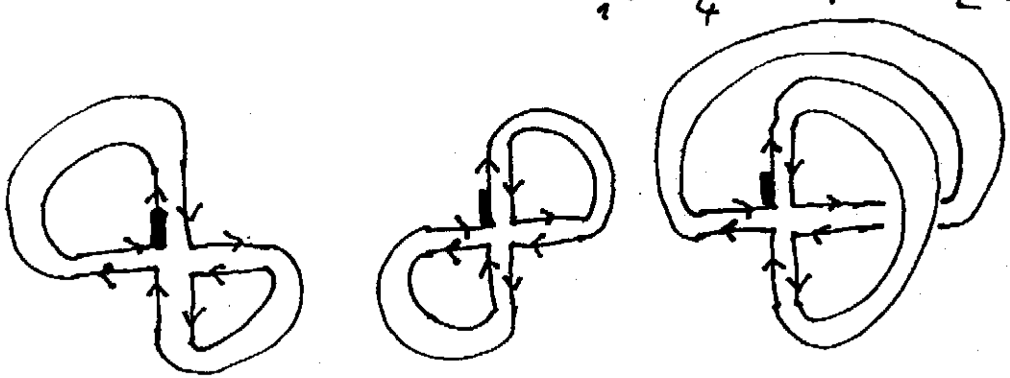
$l = i$
$k = j$

$$f(M) = \mathrm{Tr}(M^4) = \sum_{1\le i_1,\dots,i_4\le N} M_{i_1i_2}\,M_{i_2i_3}\,M_{i_3i_4}\,M_{i_4i_1}$$

$$M_{i_1i_2}M_{i_2i_3}M_{i_3i_4}M_{i_4i_1} \approx$$



$$\langle \mathrm{Tr}(M^4)\rangle = \sum_{i_1\dots i_4}\ \sum_{P}\ \prod_{\substack{[(i_k i_{k+1})(i_l i_{l+1})]\\ \Uparrow \\ P}} \frac{1}{N}\,\Delta_{i_k i_{l+1}}\,\Delta_{i_l i_{k+1}}$$

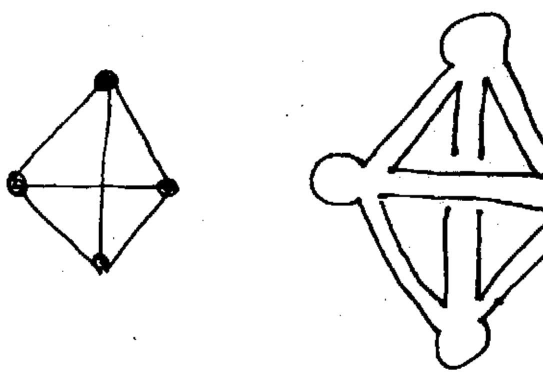


$$\langle \mathrm{Tr}(M^4)\rangle = \sum_{F} N^{p(F)-e(F)}$$

$p(F)$ = # directed cycles

$e(F) = |P| = \frac{4}{2} = 2$

$\boxed{\text{Map} = \text{Fatgraph}}$ a graph together with fixed cyclic ordering of the incident edges of each vertex: this defines embedding on an orientable 2D surface.



F

<u>a face</u>: each component of the boundary

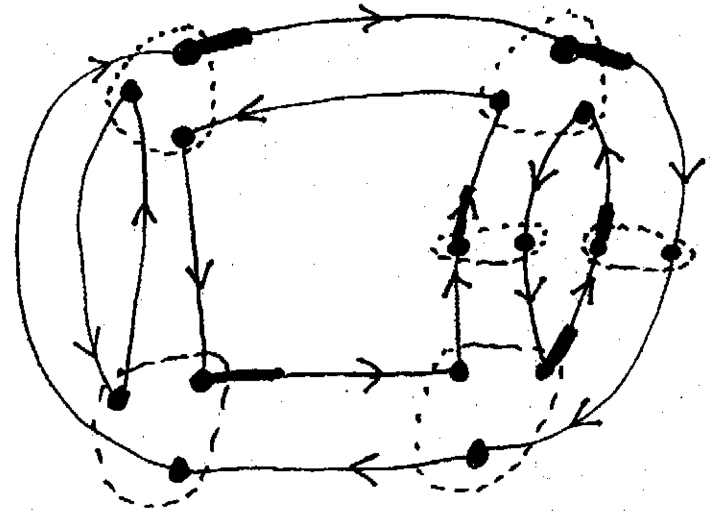
$p(F) = \#$ faces

Euler fla : $2g(F) = 2c(F) + e(F) - v(F) - p(F)$

$$f(M) = \left[\mathrm{Tr}(M^3)\right]^4 \left[\mathrm{Tr}(M^2)\right]^3$$

$$\langle f(M) \rangle = \sum_F N^{p(F)-e(F)}$$

F pointed fatgraph with
4 vertices of degree 3 and
3 vertices of degree 2



From matrices to directed graphs. $M_{ij} \approx \underset{i}{\bullet} \xrightarrow{M_{ij}} \underset{j}{\bullet}$

$D = (N, N \times N)$ complete directed

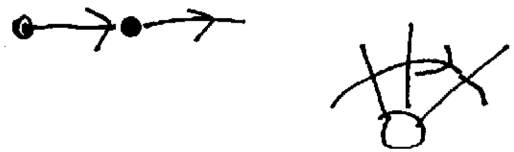
$$f(M) = [\mathrm{Tr}(M^3)]^4 [\mathrm{Tr}(M^2)]^3 = \Big(\sum_{p_1} \prod_{e \in p_1} M_e \Big)^4 \Big(\sum_{p_2} \prod_{e \in p_2} M_e \Big)^3$$

$$= \sum_{q = q_1 q_2 \cdots q_7} \prod_{e \in q} M_e$$

q is disjoint union of 7 pointed directed closed walks, 4 of length 3 and 3 of length 2.

q is given together with its decomposition into $q_1, \dots, q_7$

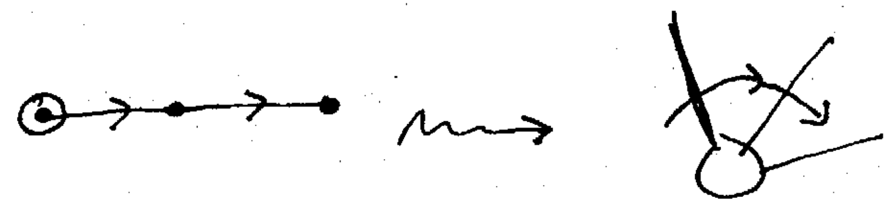
walks



Main question

Can enumeration of <u>embeddable graphs</u> be expressed as a matrix integral?

The trick: <u>pointed walks</u>



$A \subseteq E(D)$ is <u>Eulerian</u> if $\mathrm{indeg}_A(v) = \mathrm{outdeg}_A(v)$ for each $v \in V(D) = \{1, \dots, N\}$.

$$G(M, Nz\gamma^{-1}, \gamma) = \sum_{A \text{ eul.}} \Big(\prod_{e \in A} M_e\Big) \prod_{C \text{ component of } A} \frac{(Nz)^{|C|/2+1} - Nz\gamma^{|C|/2}}{Nz - \gamma}$$

Ice-type partition fction

Gimenez, Noy 2005

$p(n) \sim c\, n^{-7/2} \gamma^{n}\, n!$ $\quad c \doteq 0.426, \ \gamma \doteq 27.2$

<u>Theorem</u> (K, L)

$$\exp\Big(\sum_{n\geq 1}\sum_{r\geq 0}\big[p(n,r)+e_1(N,n,r)\big]\frac{r^{r}\,\gamma^{n-2}}{n!}\ \text{`}\leq_{\rho}\text{'}$$

$$e^{-N^2}\langle G(M, Nr\gamma^{-1}, \gamma)\rangle$$

$$\text{`}\leq_{N/2}\text{'} \exp\Big(\sum_{n\geq 1}\sum_{r\geq 0}\big[p(n,r)+e_2(N,n,r)\big]\frac{r^{r}\,\gamma^{n-2}}{n!}$$

$p(n,r)$ = # labelled connected simple graphs on n vertices which have planar embedding with r faces.

$e_1, e_2 = O(N^{-1})$

'$\leq$' as formal power series

'$\leq_{\rho}$' coef. $k \leq \rho(N) \nearrow \infty$ slower than N

Formal Power Series versus convergence

classic fla (Brézin ...)

$$f(M) = e^{(-N)\sum_{i\geq 1} t_i \operatorname{Tr}\left(\frac{M^i}{i}\right)}$$

$$\log \langle f \rangle \ \text{'='}\ \sum_{g\geq 0} N^{2-2g} \sum_{(n_1 \ldots n_k)\in \mathbb{N}^k} \prod_{i=1}^{k} \frac{(-t_i)^{n_i}}{n_i!} M_g(n_1, \ldots, n_k)$$

$M_g(n_1, \ldots, n_k)$ is the # of <u>connected maps</u> of genus g, n_i vertices of degree i.

Guionnet (Proceedings ICM 2006 Madrid)

converges for $\bar{f}(M) = e^{(-N)\sum_{i=1}^{k} t_i \operatorname{Tr}(A^i)}$

for any finite set $\bar{I} = \{p_1, \ldots, p_k\}$.

Ercolani, Mclaughlin, Guionnet, Maurel-Segala (2002)

<u>Theorem</u>. Let $\mathcal{A}$ be set of all pairs (q, K) where q is a subset of $E(D) = N \times N$ and K is a decomposition of q into directed cycles of length ≥ 3. Then

$$\left\langle \sum_{(q,K)\in\mathcal{A}} \prod_{e\in q} M_e \right\rangle = \sum_{[(G,C)]} \frac{N(N-1)\cdots(N-V(G)+1)}{|\mathrm{Aut}\,(G,C)|\, N^{e(G)}}$$

all isomorphism classes of pairs (G, C) where G is a simple graph with $\leq N$ vertices and no isolated vertex, C is a specified <u>DCDC</u> of G.

! $\sum_{\mathcal{A}} \prod_{e\in q} M_e$ is an Ihara-Selberg-type function !