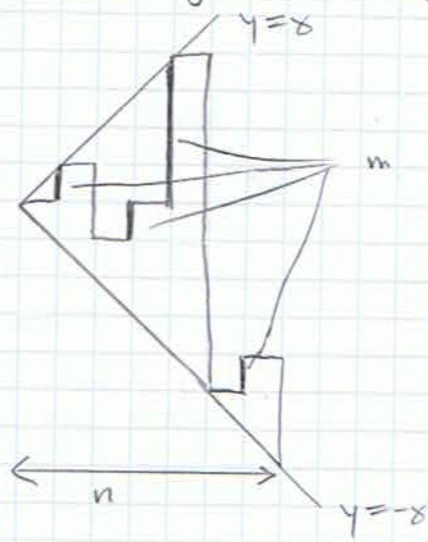
Partially directed walks in a symmetric wedge

(and crossings of matchings)

T. Prellberg (QMUL), Newton Institute, April 2008.

1.) Partially directed self-avoiding walks in $|y| \le x$ (PDSAW)



\# of up-steps : m

\# of right-steps : n

(\# of down-steps : $n+m$)

total \# of walks for n fixed: $(2n-1)!!$

Proposition 1 (Janse van Rensburg, Lechnitzer, Prellberg, 2006)

Let $c_{n,m}$ be the \# of PDSAW in the wedge $|y| \le x$

starting at $(0,0)$ and ending at $(n,-n)$ with m up-steps.

The generating function $G(x,y) = \sum_{n,m=0}^{\infty} c_{n,m} x^n y^m$

is given by

$$G(x,y) = C\left(\frac{x}{1-y}\right) \sum_{k=0}^{\infty} (-1)^k y^{\binom{k+1}{2}} \left[C\left(\frac{x}{1-y}\right) - 1\right]^k$$

where $C(t) = \dfrac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} \dfrac{t^n}{n+1}\binom{2n}{n}$

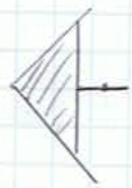
Proof: Functional equation + Kernel method

Consider PDSAW ending in horizontal step at height n_a

$$G(x,y;a,b) = \sum' C_{n,m}^{n_a,n_b}\, x^n y^m a^{n_a} b^{n_b}$$

$$g(a,b) = G(x,y;a,b)$$

$$g(a,b) = 1 + xab\, g(a,b) + xab \frac{y\, b/a}{1-y\, b/a}\, g(a,b)$$

  +  + 

$$+ xab \frac{y\, a/b}{1-y\, a/b}\, g(a,b) - xab \frac{y\, a/b}{1-y\, a/b}\, g(a,ay)$$

\+  − 

$$- xab \frac{y\, b/a}{1-y\, b/a}\, g(by,b)$$

− 

$$\rightsquigarrow \quad K(a,b)\, g(a,b) = 1 - F(a,b)\, g(a,ay) - F(b,a)\, g(by,b)$$

- Set the <u>kernel</u> $K(a,b)=0$ fixes $b=b_{1,2}(a)$

- solve $1 = F(a,b)\, g(a, ay) + F(b,a)\, g(by, b)$

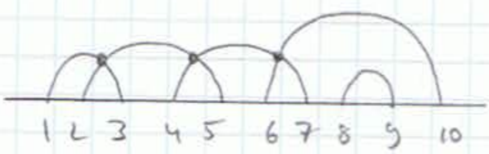
  by iteration:

$$\xleftarrow{} \quad a_{-1}=b_2(a) \xleftarrow{} a_0 = a \xrightarrow{} a_1 = b_1(a) \xrightarrow{} a_2 = b_1(b_1(a))$$

$(b_1 \circ b_2 = b_2 \circ b_1 = id)$

- $G(x,y) = G(x,y;\ a \leftarrow y,\ b \leftarrow 1)$ gives the claimed result $\square$

2. Nestings and crossings of matchings



matching of a $2n$-set $(n=5)$

with 3 crossings and 1 nesting.

<u>Fact (not obvious yet)</u>: the statistics on crossings and nestings is symmetric

<u>Proposition 2 (Touchard - Riordan formula)</u>

Let $M_{n,m}$ be the # of matchings of a $2n$-set with m crossings. Then

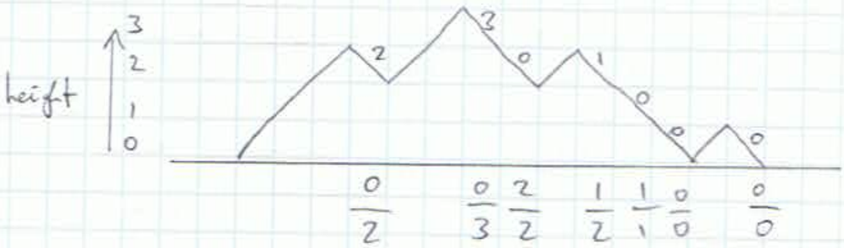
$$\phi_n(q) = \sum_{m=0}^{\infty} M_{n,m}\, q^m = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{\binom{k+1}{2}} \binom{2n}{n+k}$$

First proof by Touchard (1954).

Theorem 3 $\boxed{C_{n,m} = M_{n,m}}$

Proof: $$G(x,y) = \sum_{n=0}^{\infty} x^n \phi_n(y)$$

3. Weighted Dyck paths



Weight $0 \leq w_i \leq h$

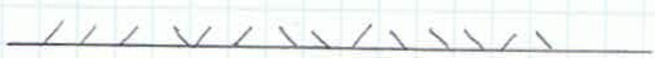
Complementary weight $w_i^c = h - w_i$

- weights on down-steps, bdd by height
- weight of path $= \sum$ of weights

Proposition 4 (de Sainte-Catherine, 1983) Matchings of a $2n$-set with C crossings and N nestings are in bijection with Dyck paths with n down steps, weight C and complementary weight N

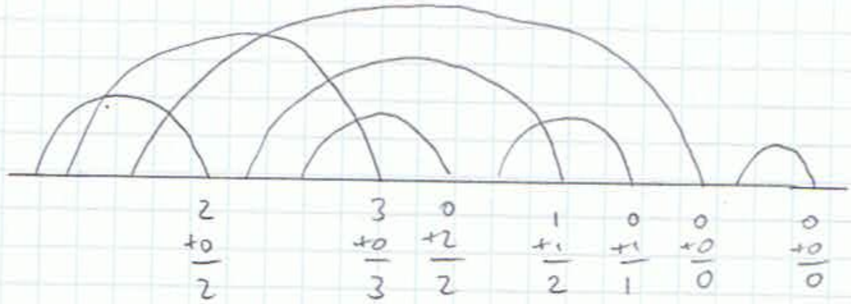
Proof Sketch of an example bijection

- up/down steps encode beginning/end of arches



- weight encodes # of connections missed

(this generates crossings / nestings)



crossings ≙ weight
nestings ≙ compl weight

□

<u>Corollary 5</u> The statistics of crossings and nestings is symmetric

4. Continued fraction expansion

<u>Proposition 6</u>

$$G(x,y) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{(1+y)\,x}{1 - \cfrac{(1+y+y^2)\,x}{1 - \dots}}}}$$

<u>Proof</u> $G_h(x,y)$ GF for weighted Dyck paths at height h

Necklace: $G_h(x,y) = \left[1 - x\,(1+y+y^2+\dots+y^h)\, G_{h+1}(x,y)\right]^{-1}$

and $G(x,y) = G_0(x,y)$ □

5. Two bijective proofs of Theorem 3

(posed as open problem at FPSAC 2007)

- Rubey, Dec 2007: bij to weighted Dyck paths to nestings
- Poznanović, March 2008: bij to some statistics on matchings to nestings