Notes on Combinatorial Zeta-function

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prime reduced cycle, Ihara-Selberg function, Zeta function, Bass theme, Mac Mahon Master theme, non-commutative, colored Jones function, q-chromatic function, U-polynomial, weight systems, Feynman-Kac-Sherman-Ward solution

2D Ising, Witt identity
$G = (V, E)$ graph with arbitrary orientation

walk

$\mu = v_1a_1v_2 \ldots v_{a+1} = v_1$

$\alpha_e \in \{a_e^1, a_e^2 \mid e \in E\}$

$\alpha_e \neq a_{e+1}^1$

$(a_1 \ldots a_m) \neq 2^m$, if $m > 1$

Thara–Selberg function: $I(n) = \prod \left(1 - \frac{1}{\omega^m}\right)$

Zeta function: $\zeta(n) = \frac{1}{I(n)}$
Bass Theorem. For any graph $G$,\n\[
I(m) = \det(I - mT)
\]

MacMahon Master Theorem: A matrix $(m \times m)$, \[x = (x_1, \ldots, x_m)\] commuting variables.

Then the coeff. of $x_1^{m_1} \cdots x_m^{m_m}$ in $\prod (\sum_{i=1}^{m} a_{ij} x_j)^{m_i}$ is equal to the coeff. of $x_1^{m_1} \cdots x_m^{m_m}$ in the power series expansion of $[\det(I - XA)]^{-1}$ \[X = \left( \begin{array}{c} x_1 + z_1 x_2 \\ 0 \\ \vdots \\ 0 \\ x_m \end{array} \right)\]

$T$: matrix of transitions between edges \[a, a' \in \{a_e, \bar{a}_e : e \in E\} \quad a \rightarrow a'\]

Equivalently:\n
\[
\frac{1}{\det(I - B)} = \sum_{\text{flow}} (B(f)) \text{mult}(f)
\]

$B(f) = \prod_{e \in f} B(e) f(e)$ \[\text{mult}(f) = \prod_{e \in f} \text{mult}_e(f)\] \[\text{mult}_e(f) = \langle f(e^-), f(e) \rangle\]
Lyndon word: prime, minimal in its cyclic rearrangements
(\(X\) lin. ordered finite, \(X^*\) lexicogr. ordered)

Then (Lyndon)
\[X^* \in \mathcal{L} = l_1 \ldots l_m, \ l_i \succ l_{i+1}, \text{ unique decomposition}\]

\((b_{i,j}) = \beta : X \times X \text{ matrix of commuting variables}\]
\[b_{i,j} : \text{weight of transition between } i, j\]

\(X\): vertices of digraph \(\rightarrow\) Mac Mahon Master equation
\(X\): edges of digraph \(\rightarrow\) Bass theorem
\( (b_{i,j}) = B \) \; \text{and} \; w = x_1 x_2 \ldots x_m \in X^* \; \text{where} \; w = l_1 \ldots l_m \n \text{Beirc} (w) = b(x_1 x_2) b(x_2 x_3) \ldots b(x_m x_1) \n \text{Beirc} (w) = \text{Beirc} (l_1) \ldots \text{Beirc} (l_m) \n \omega' = x_1' x_2' \ldots x_m' \; : \; \omega \text{ in non-decreasing order} \n \text{Beirc} (w) = b(x_1' x_2) b(x_2' x_3) \ldots b(x_m' x_m) \n \text{Then:} \; \prod_{k \in \mathcal{K}} (I - \text{Beirc} (b))^{-1} = \sum_{\omega \in X^*} \text{Beirc} (\omega) = \sum_{\omega \in X^*} \text{Beirc} (\omega) \n \left( \prod_{k \in \mathcal{K}} \frac{1}{\text{det}(I - B)} \right) = \text{Beirc} (I - B) \n \text{MMMT.} \n \text{There is a non-commutative MMMT. (?) What about Beirc (?)}
Non-commutative MacMahon Master Th. (Garoufalidis, L., Zeilberger)

Important for knots.

Thm (Lebl, Garoufalidis, Math. Annalen 2006)

Colored Jones function is "an evaluation of \( \det_q(I-B) \)

\[
\det_q(A) = \sum_{\pi} (-q)^{\# c: \pi(c) \neq c} \prod a_{\pi(c)}
\]

\( \frac{1}{\text{det}_q(I-B)} \) an integral?
Colored Jones function of knot $K$ is a power series whose coefficients are equal to "defected" $q$-chromatic function applied to linear combination of long chord diagrams constructed from flows on reduced $K$.

Is there a relation with the weight systems expression for the colored Jones?

Other connections of chromatic polynomial and knot pol...

1. Chromatic weight systems (Chmutov, Duzhn, Lado, Noble, Weld...)
2. Homology (Lecot, Moffett, Advances in Math 2009)
$q$-chromatic function

$$M_{q^v}(G, k) = \sum_{v \in V} q^{\chi_0(v)}$$

$$M_{q^v}(K_{m-1}, k) = m! \cdot \binom{k}{m} \cdot \prod_{i=1}^{m-1} q^i$$

$$\binom{k}{m} = \frac{q^m}{q-1}$$

$$M_{q^v}(G, k) = \sum_{A \subseteq E} (-1)^{|A|} \prod_{w \in C(A)} q^{|V| - |A|}$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$\sum_{w \in E} \prod_{\{u_1, \ldots, u_{k-1}\}} \left[ \prod_{i=1}^{k-1} \left( \sum_{v \in V} \chi_0(v) \right) \right]$$

$$E(p_k)(v) = \sum_{w \in E} \left( \chi_0(w) \chi_0(v) \right)$$

$q$-dichromatic

$q$-Izing

$q$-Izing?
How strong is 9-dichromate?

\[ U_G(\ell_1, x_1, \ldots, x_i, \ldots) = \sum_{A \in E(G)} x(T_A) (\ell - 1) \]

\[ \ell_A = (m_1, \ldots, m_k) \text{ partition determined by } G - A \]

\[ x(T_A) = x_{m_1} \cdots x_{m_k} \]

Conj. (Lebel 2007) By determines \( U \)

Conj.

\[ \ell = (m_1, \ldots, m_k) \text{ partition of } n \]

\[ c(\ell) = \left[ c(\ell) \right]_{\ell = 0, 1, \ldots, i, \ldots} \]

\[ c(\ell)_{i_j} = \prod_{j=1}^{k} \left( m_j \ell^\gamma + m_j \ell^{\gamma-1} \right) \]

\( \) Only trivial rational linear comb. of \( c(\ell) \)'s is identically zero \( \)
Feynman, Potts, Sherman, Ward solution to 2D Ising

\( G = (V, E) \) planar map

\( W(\mu) = (-1)^{\text{rot}(\mu)} \prod_{\alpha \epsilon p} \text{rot}(\alpha, \alpha') \)

\( \mu \): aperiodic closed walk

\( \mu \) can be written as \( \prod_{\alpha, \alpha'} \text{rot}(\alpha, \alpha') \)

Theorem. \( \chi(G, x) = \prod_{\alpha \epsilon p} (1 - W[\mu]) \)

This gives with Bass: \( \chi(G, x) \) contains rotations

\( \chi^2(G, x) = \text{det} (I - \tilde{\tau}) \)

\( \text{Pfaff} (A_D) = \sqrt{\text{det} (I - \tilde{\tau})} \) combinatorially
In the proof of FPSW theorem, Will identity plays crucial role.

\[ \prod (1 - Z_1^{m_1} \cdots Z_k^{m_k})^M(m_1, \ldots, m_k) = 1 - R_1 - R_2 - \cdots - R_k \]

Let \( R_1, \ldots, R_k \) be commuting variables. Then

\[ M(\cdot) = \# \text{ different non-periodic circular sequences made from the roll of } m_i \text{ variables } R_i, \ i = 1, \ldots, k. \]

\[ \text{?} \ Is \ there \ Will \ in \ Bass \? \]