Non-universality of the Kármán "Constant"

by

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Traditionally the Kármán constant is estimated from the overlap region (log-layer) alone, which leads to a great deal of uncertainty, especially at lower $Re$, and the various approaches used often incorporate assumptions that have not been fully validated (e.g., limits of overlap region).

The uncertainty can be greatly reduced with the help of empirically-based "composite profiles" for the various wall-bounded flows, that are carefully constructed to satisfy several constraints.
ZPG TBLs - Results
The composite form can now be used to fit experimental data to determine, $u_\tau$, $\Pi$ and $\delta$. 
\[
\frac{dU_{inner}^+}{dy^+} = P_{23} + P_{25}, \quad \Xi = y^+ \frac{dU^+}{dy^+}
\]

\[
P_{23} = b_0 \frac{1 + b_1 y^+ + b_2 y^{+2}}{1 + b_1 y^+ + b_2 y^{+2} + \kappa b_0 b_2 y^{+3}}
\]

\[
b_0 = 1.000 \times 10^{-2} \kappa^{-1} \quad (\kappa = 0.384)
\]

\[
b_1 = 1.100 \times 10^{-2}
\]

\[
b_2 = 1.100 \times 10^{-4}
\]

\[
P_{25} = (1 - b_0) \frac{1 + h_1 y^+ + h_2 y^{+2}}{1 + h_1 y^+ + h_2 y^{+2} + h_3 y^{+3} + h_4 y^{+4} + h_5 y^{+5}}
\]

\[
h_1 = -1.000 \times 10^{-2}
\]

\[
h_2 = 6.000 \times 10^{-3}
\]

\[
h_3 = 9.977 \times 10^{-4}
\]

\[
h_4 = 2.200 \times 10^{-5}
\]

\[
h_5 = 1.000 \times 10^{-6}.
\]

\(\Xi_{23} + \Xi_{25} \) (---); \(\Xi_{23} \) (--- ---); \(\Xi_{25} \) (--- --- ---); KTH (\(\triangle\)); NDF (○); Carlier & Stanislas (2005) (×); T. Nickels et al., 2007(□).

Comparison of new fit \(\Xi_{23} + \Xi_{25} \) (---) with previous proposals in the literature. Other functional forms shown for \(\kappa = 0.384\) and \(B = 4.17\) are: ---×---, Musker (1979); *, Modified Musker from Chauhan, Monkewitz, & Nagib, 2007; -- -- -- --, Spalding (1961); · · · · · ·, Van Driest from White (2006); · · · · · · · · · ·, Gersten & Herwig from Schlichting & Gersten (2000).
Monkewitz, Chauhan, & Nagib, 2007, *Physics of fluids*

\[ \eta = \frac{y}{\Delta} \]

\[ U_{\infty}^+ - U_{outer}^+ = \left[ \frac{1}{\kappa} E_1(\eta) + w_0 \right] \frac{1}{2} \left[ 1 - \tanh\left( \frac{w_{-1}}{\eta} + w_2 \eta^2 + w_8 \eta^8 \right) \right] \]

\[
\begin{align*}
    w_0 &= 0.6332 \\
    w_{-1} &= -0.096 \\
    w_2 &= 28.5 \\
    w_8 &= 33000.
\end{align*}
\]

\[ E_1(\eta) = \int_{\eta}^{\infty} \frac{\exp^{-t}}{t} dt = -\gamma - \ln(\eta) + \eta + O(\eta^2) \quad \text{for} \quad \eta \ll 1 \]

\[
\sim \frac{\exp^{-\eta}}{\eta} \left[ 1 - \frac{1}{\eta} + O(\eta^{-2}) \right] \quad \text{for} \quad \eta \to \infty ,
\]

\[ \gamma = 0.57721566 \quad \text{Euler-Mascheroni constant} \]


\[
\int_0^\infty W^+ d\eta \equiv 1,
\]
Consistent with over 300 velocity profiles from some 35 experiments with the same log-law parameters $\kappa = 0.384$, $B = 4.17$ and $C = 3.3$

Contrast to "traditional" values $\kappa = 0.41$, and $B = 5.0$
FIG. 1: (Color) Comparison of $U_\infty^+$ and shape factor, $H$ with the logarithmic theory. (a) $U_\infty^+$ versus $Re_\delta^*$. $U_\infty^+$ for experiments obtained using the power law relation (B7) and shown in red symbols. Solid black symbols are direct oil-film measurements from KTH and NDF. (b) $H$ versus $Re_\delta^*$. Shape factor for the theory obtained by integration of logarithmic composite profile. ‘Blue — — ’, Prediction of logarithmic theory, and ‘Cyan — — ’ with low $Re$ terms of Chauhan et al.\textsuperscript{8}. Data points shown for various experiments are represented as (listed alphabetically) : □, Bell; ◊, Bruns\textsuperscript{17}; ⊕, Carlier & Stanislas\textsuperscript{18}; ⊙, DeGraff & Eaton\textsuperscript{19}; +, Erm\textsuperscript{20}; ▽, Hites\textsuperscript{12}; ◄, Karlsson\textsuperscript{21}; ■, Knobloch & Fernholz\textsuperscript{13}; ◊, Nagib et al.\textsuperscript{14,22}; □, Naguib\textsuperscript{23}; ⊗, Nickels et al.\textsuperscript{24}; △, Österlund\textsuperscript{6}; ⊙, Purtell et al.\textsuperscript{25}; ◿, Smith\textsuperscript{7}; ×, Smith & Walker\textsuperscript{26}; ⊞, Wark\textsuperscript{27}; *, Wieghardt. Additional symbols in (b) represent : *, data of Petrie et al. from Fernholz & Finley\textsuperscript{28}; ⊙, Priyadarshana & Klewicki\textsuperscript{29}; ⊙, Spalart\textsuperscript{30}; Data of Bell and Wieghardt obtained from Coles\textsuperscript{31}. 
Channel & Pipe Flows
Channels & Pipes The composite profile

Channels & Pipes:

\[-\frac{\bar{w}}{u^2} + \frac{dU^+}{dy^+} = 1 - \frac{y^+}{Re_\tau}\]

\[
\frac{dU^+_{\text{inner}}}{dy^+} = \frac{1}{s} + \frac{y^{+2}}{\kappa} - \frac{y^+}{s Re_\tau}
\]

\[
\frac{1}{s} + \frac{y^{+2}}{\kappa} + y^+^3
\]

\[U^+ = U^+_{\text{inner}} + \frac{2\Pi}{\kappa} W(\eta), \quad \eta \equiv y/H \leq 1\]

\[
\left.\frac{d^nU^+}{dy^+^n}\right|_{\eta=1} = 0 \quad n \text{ is odd}
\]

\[
\neq 0 \quad n \text{ is even}
\]

\[
W_{\text{pipe}} = \frac{1 - \exp\left[-\frac{1}{3}(4p_2 + 6p_3 + 7p_4)\eta^3 + p_2\eta^4 + p_3\eta^6 + p_4\eta^7\right]}{1 - \exp\left[-(p_2 + 3p_3 + 4p_4)/3\right]} \left(1 - \frac{\ln(\eta)}{2\Pi}\right)
\]

where, \(p_2 = 4.075, \quad p_3 = -6.911\) and \(p_4 = 4.876\).

\[
W_{\text{channel}} = \frac{1 - \exp\left[-\frac{1}{2}(3c_2 + 6c_3 + 7c_4)\eta^2 + c_2\eta^3 + c_3\eta^6 + c_4\eta^7\right]}{1 - \exp\left[-(c_2 + 4c_3 + 5c_4)/2\right]} \left(1 - \frac{\ln(\eta)}{2\Pi}\right)
\]

where, \(c_2 = -20.22, \quad c_3 = 17.1\) and \(c_4 = -11.17\).

The composite profiles for Channels and Pipes fit accurately to both DNS and experimental data. Fit provides the local \(\kappa, B\) and \(\Pi\).
Near the wall DNS of various flows show a cubic behavior for Reynolds shear stress, $-\overline{uv}$.

However, differences between DNS of ZPG TBLs by Spalart (1988) and Skote (2001) are observed for $1 - \frac{dU^+}{dy^+}$.

We have also found differences at higher $y^+$ for the gradient of total stress for these two DNS.
\( \Psi_i \) is almost constant across \( y \) for \( \kappa = 0.41 \) but the magnitude of \( \Psi_i \) decreases with increasing Reynolds number.

\( \Psi_i \) for \( \kappa = 0.422 \) is has a very good constant behavior for \( y^+ > 600 \). \( \Psi_o \) is clearly constant across \( y/R \) for \( \kappa = 0.41 \), except the data with \( Re_\tau \lesssim 11,000 \).

\[ \Psi_i = U^+ - \frac{1}{\kappa} \ln(y^+) \equiv B \]

\[ \Psi_o = (U_c^+ - U^+) + \frac{1}{\kappa} \ln(y/R) \equiv -B_o \]

\[ 300 < y^+ < 0.15 Re_\tau \]

Comment on consistency, and single \( \kappa \) for all \( Re \) versus composite profile approach yielding \( \kappa(Re) \).
\[ \delta^+ = R e_\tau = 500 \]
— ZPG TBL \( \kappa=0.384, B=4.127, \Pi=0.12941 \)
— Channel \( \kappa=0.37, B=3.7, \Pi=0.05 \)
— Pipe \( \kappa=0.41, B=5.0, \Pi=0.3 \)
\[ \delta^+ = Re_\tau = 1000 \]

--- ZPG TBL \( \kappa=0.384, B=4.127, \Pi=0.34267 \)

Channel \( \kappa=0.37, B=3.7, \Pi=0.05 \)

Pipe \( \kappa=0.41, B=5.0, \Pi=0.3 \)
\[ \delta^+ = Re_\tau = 2000 \]

- ZPG TBL \( \kappa=0.384, B=4.127, \Pi=0.40383 \)
- Channel \( \kappa=0.37, B=3.7, \Pi=0.05 \)
- Pipe \( \kappa=0.41, B=5.0, \Pi=0.3 \)
\[ \delta^+ = Re_\tau = 10000 \]

--- ZPG TBL \( \kappa=0.384, B=4.127, \Pi=0.44054 \)

Channel \( \kappa=0.37, B=3.7, \Pi=0.05 \)

Pipe \( \kappa=0.41, B=5.0, \Pi=0.3 \)
\[ \frac{u' \nu'}{u^2} \]
The wake parameter, $\Pi$, for pipe flows is higher than that for channel flows, although they have the same governing equation and boundary condition.

Parallel flows do not show the development of $\Pi$ at low Reynolds numbers.

![Graph showing the relationship between $\Pi$ and $Re_\delta^*$]
\[ \tilde{\kappa} \tilde{B} = 1.6 \left[ \exp(0.1663 \tilde{B}) - 1 \right] \]
Overlap parameters for wall-bounded flows, including the von Kármán "coefficient," are not universal.

They appear to at least depend on pressure gradient and flow geometry.

However, their asymptotic high Reynolds number values are different constants for fully developed or near equilibrium flows.

Full consequences on currently popular models for prediction of turbulence should be fully assessed.
The 19 data points that are fitted to get $\kappa = 0.4265$ are within $\pm 2\sigma$ ($\sigma$=standard deviation), while the 11 points that fitted to get $\kappa = 0.4126$ are only within $0.8\sigma$.

Fit to the intermediate Reynolds numbers is better. Value of $\kappa = 0.4126$ is again in consistent agreement with those observed in previously.

As the experimental Reynolds number increases, the accuracy required in obtaining wall shear stress and hence $U_c^+$ needs to increase too.

\[
U_c^+ = \frac{1}{\kappa} \ln \left( \frac{R u_\tau}{\nu} \right) + B - B_o
\]

\[
B - B_o \equiv U_c^+ - \frac{1}{\kappa} \ln \left( \frac{R u_\tau}{\nu} \right)
\]
Other functional forms for ZPG TBLs
At least two length and velocity scales needed

**Inner**: Viscous effects are dominant

\[
\frac{U}{U_{si}} = F_{\text{inner}}(y/L_{si}, Re, *)
\]

**Outer**: Inertial effects are dominant. Effects of viscosity are negligible

\[
\frac{U_{\infty} - U}{U_{so}} = F_{\text{outer}}(y/L_{so}, Re, *)
\]

**Matching**: \( U^+ = \frac{U_{si}}{u_\tau} \left[ \lim_{y/L_{si} \to \infty} F_{\text{inner}} \right] + \frac{U_{so}}{u_\tau} \left[ \lim_{y/L_{so} \to 0} F_{\text{outer}} \right] = F(Re) \)

Also, scaling velocity for the Reynolds stresses, \( u'_i u'_j, (R_{si}, R_{so}) \) are needed.

**Equilibrium**: Self-similarity of mean flow & Self-preservation of turbulence

<table>
<thead>
<tr>
<th></th>
<th>( U_{si} )</th>
<th>( L_{si} )</th>
<th>( U_{so} )</th>
<th>( L_{so} )</th>
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<tbody>
<tr>
<td>Classical</td>
<td>( u_\tau )</td>
<td>( \nu/u_\tau )</td>
<td>( u_\tau )</td>
<td>( \delta, \Delta )</td>
</tr>
<tr>
<td>George &amp; Castillo (997)</td>
<td>( u_\tau )</td>
<td>( \nu/u_\tau )</td>
<td>( U_\infty )</td>
<td>( \delta_{99} )</td>
</tr>
<tr>
<td>Zagarola &amp; Smits (998)</td>
<td>( u_\tau )</td>
<td>( \nu/u_\tau )</td>
<td>( U_\infty \delta^*/\delta )</td>
<td>( \delta )</td>
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Completely self-consistent description of ZPG TBLs based on the classical (logarithmic) theory recently established by Monkewitz et al., 2007 and Chauhan, Monkewitz, & Nagib, 2007
Mean flow similarity George & Castillo (1997)

With original coefficients

With modified coefficients
\[
\frac{1}{H} = 1 - U_{\infty}^{-1} \int_{0}^{\infty} W^{+2} \, d\eta
\]

\[
\int_{0}^{\infty} W^{+2} \, d\eta \equiv I_{WW} = 7.1075 + O\left(\frac{\ln(Re_\delta^*)}{Re_\delta^*}, \frac{1}{Re_\delta^*}, \ldots\right)
\]

\[
H(Re_\delta^*) = \frac{\kappa^{-1} \ln(Re_\delta^*) + C}{\kappa^{-1} \ln(Re_\delta^*) + C - I_{WW}}
\]

\[
U_{\infty}^+ = \frac{1}{\kappa} \left[ \ln(Re_\theta) + \ln(H) \right] + C
\]

\[
H(Re_\theta) \sim 1 + \frac{\kappa I_{WW}}{\ln(Re_\theta)} + \frac{\kappa^2 I_{WW} (I_{WW} - C)}{\ln^2(Re_\theta)} + \frac{\kappa^2 I_{WW} (\kappa I_{WW}^2 - I_{WW} - 2\kappa I_{WW} C + \kappa C^2)}{\ln^3(Re_\theta)} + \ldots + O\left(\frac{1}{Re_\theta}\right)
\]

\[
\ln(H) \sim \frac{\kappa I_{WW}}{\ln(Re_\theta)} + \frac{\kappa^2 I_{WW} (I_{WW} - 2C)}{2 \ln^2(Re_\theta)} + \ldots + O\left(\frac{1}{Re_\theta}\right)
\]

\[
\kappa = 0.384, \quad C = 3.3, \quad I_{WW} = 7.11
\]

(a) Shape factor \(H\) versus \(Re_\theta\). KTH(\(\triangle\)); NDF (○); (-----) and \(H\) determined numerically from additive (- - - -) and multiplicative (- ⋅ - ⋅ -) composite profiles. (b) \(\ln(H)/\kappa\) minus the leading term \(I_{WW}/\ln(Re_\theta)\) versus \(Re_\theta\). NDF oil-film (●); KTH oil-film (▲) experiments.
Shape Factor, \( H = \delta^*/\theta \)

\[
H = \frac{1}{1 - I_{WW}(u_\tau/U_\infty)}, \quad I_{WW} = \int_0^\infty (U_\infty^+-U_\infty^+)^2 d\left(\frac{y}{\Delta}\right) \approx 7.11
\]

Integral \( I_{WW} \) asymptotes to a constant in ZPG TBLs.

The shape factor for various experiments show consistent behavior and decreases with Reynolds number.

The equation for \( H \) is approximated using classical theory to agree remarkably well with experimental data.

The asymptotic limit of shape factor is predicted as unity.

\[
H^{-1} = 1 - U_\infty^+ \int_0^\infty W^+ d\eta.
\]

\[
U_\infty^+ = \frac{1}{\kappa} \ln(Re_{\delta}) + C + O(Re_{\delta}^{-1}) \quad \text{with}
\]

\[
C = -\frac{\gamma}{\kappa} + w_0 + B = 3.30.
\]
The Coles-Fernholz type relation and velocity defect in the outer part

\[
U_\infty^+ = \frac{1}{\kappa} \ln(Re_{\delta^*}) + C
\]

Inner: \( U / u_\tau = F_{\text{inner}}(y^+) \) \( \lim_{y^+ \to \infty} F_{\text{inner}} = U^+ = \frac{1}{\kappa} \ln(y^+) + B \)

Outer: \( W^+ = \frac{U_\infty - U}{u_\tau} = F_{\text{outer}}(y/\Delta) \) \( \lim_{\eta \to 0} F_{\text{outer}} = U_\infty^+ - U^+ = -\frac{1}{\kappa} \ln(y/\Delta) + C_\Delta \)

Matching: \( U_\infty^+ = \lim_{y^+ \to \infty} F_{\text{inner}} + \lim_{\eta \to 0} F_{\text{outer}} = U_\infty^+(Re_{\delta^*}) \)

For ZPG TBLs, \( \frac{\Delta}{\delta} \to \text{constant} \).

Therefore, either of \( \Delta = U_\infty^+ \delta^* \) (Rotta-Clauser Delta) or \( \delta \) (local boundary layer thickness) can be used in the outer part.

\( \kappa = 0.384, \quad B = 4.17, \quad C = 3.3 \)
\[
\frac{\tau_{\text{wall}}}{\rho} \equiv u_\tau^2 = (U_\infty)^2 \frac{d\theta}{dx} + \frac{d}{dx} \int_0^\infty (u'^2 - \bar{v}'^2) \, dy,
\]

\[
\frac{\bar{u}'^2 - \bar{v}'^2}{U_m^{2-m}(\bar{\eta})},
\]

Inner
\(m=0\)

Mixed
\(m=1\)

Outer
\(m=2\)
FIG. 7: (Color) The dependence of $\delta_{99}/\delta^*$ on $U_\infty^+$: Red ———, integrated GC97 composite profile with original coefficients, Red - - - -, integrated GC97 composite profile with modified coefficients, Blue ———, integrated logarithmic composite profile, and Cyan ——— logarithmic composite with low $Re$ terms of Chauhan et al.\textsuperscript{8}. For explanation of symbols see caption in Fig. 1.
Ratio of $\Delta$ over $\delta$ is an important parameter indicating the validity of Clauser theory.

$\Delta/\delta$ reaches its asymptotic value near 3.5 and remains a constant consistent with classical theory.

Invalidates arguments of power law theory of George & Castillo (1997)

\[
D \equiv \frac{\Delta}{\delta} = \frac{Re_{\delta^*}}{\delta^*} = \exp \left[ 2\Pi + \kappa(B - C^*) \right]
\]