Quantum transport in chaotic cavities and Selberg’s integral

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Outline

- Preliminaries
- Chaotic cavities: Scattering setup
- Statistics and Selberg’s integral
- Shot-noise and Fano factor
- Higher cumulants
- Distributions and asymptotics
- Conclusions
Time-dependent fluctuations of current:

\[ \delta I(t) = I(t) - \langle I \rangle \]

Noise spectral power:

\[ P(\omega) = 2 \int_{-\infty}^{\infty} dt \, e^{i\omega t} \left\langle \delta I(t + t_0) \delta I(t_0) \right\rangle \]

average \( \langle \cdots \rangle \) over the initial time \( t_0 \) \( \iff \) statistical average

“White” noise when \( P(\omega) \) does not depend on \( \omega \)

Noise results from fluctuations in the state occupation numbers
Two sources of current fluctuations

Zero-frequency spectral power: 
\[ P = 2 \int_{-\infty}^{\infty} dt \langle \delta I(t + t_0) \delta I(t_0) \rangle \]

**Thermal noise** (Johnson-Nyquist)
- thermal motion of electrons
- occurs in any conductor
- fluctuation – dissipation:
  \[ P = 4G k_B T \]
- nothing newer than conductance \( G = V/I \)
- vanishes at zero \( T \)

**Shot noise** (Schottky)
- random transport of electron charge \( e \)
- not all conductors exhibit shot noise
- Poisson (uncorrelated) transport:
  \[ P_{\text{pois}} = 2e \langle I \rangle = \langle G \rangle 2eV \]
- QM correlations: \( P < P_{\text{pois}} \)
- persists down to zero temperature
GaAs/Al$_x$Ga$_{1-x}$As heterostructures
2DEG with high mobility
QPCs in series define cavities
ballistic (mean free path $\sim$ cavity size)
Quantization of QPC’s conductance:
$N_{L,R}$: propagating modes (channels)
$T_p \leq 1$: transmission probabilities

After Oberholzer et al.: PRL 86, 2114 (2001)
GaAs/Al\textsubscript{x}Ga\textsubscript{1-x}As heterostructures

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Conductance: \[ g \equiv \frac{G}{G_0} = \sum_p T_p , \quad G_0 = \frac{2e^2}{h} \]

Landauer: '59

Shot noise: \[ p \equiv \frac{P}{P_0} = \sum_p T_p (1 - T_p) , \quad P_0 = 2e|V|G_0 \]

Lesovik: '89

Büttiker: '90

$\left\langle p \right\rangle < \sum_p \left\langle T_p \right\rangle = p_{\text{pois}}$

$\rightarrow$ noise suppression

After Oberholzer et al.: PRL 86, 2114 (2001)
Fluctuations of $T_p$ depend on conductor’s type  \text{(RMT for chaotic cavities)}

Exact expressions for $\langle g \rangle$ and $\text{var}(g)$ are well known \text{Baranger & Mello: ’94; Beenakker: ’97}

Shot noise in chaotic cavities in the “classical” regime $N_{L,R} \gg 1$:

$$\frac{\langle p \rangle}{p_{\text{pois}}} = \frac{N_L N_R}{(N_L + N_R)^2}$$ \text{Nazarov: ’95}
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Nazarov: ’95

- What are...
  - RMT value of $\langle p \rangle$ at arbitrary $N_{L,R}$ (quantum regime)?
  - higher cumulants of $g$ and $p$ at arbitrary $N_{L,R}$?
  - distributions of $g$ and $p$ at arbitrary $N_{L,R}$?
Scattering setup

S-matrix: \[
\begin{pmatrix}
b_L \\
b_R
\end{pmatrix}
= S
\begin{pmatrix}
a_L \\
a_R
\end{pmatrix}
\]

Parametrisation: 
\[
S = \begin{pmatrix}
r & t \\
t' & r'
\end{pmatrix}
\]

- \(r\) and \(r'\) — \(N_L \times N_L\) and \(N_R \times N_R\) reflection matrices
- \(t\) and \(t'\) — \(N_L \times N_R\) and \(N_R \times N_L\) transmission matrices

\(T_p, p = 1, \ldots, n,\) are eigenvalues of \(tt^\dagger\) (or \(t^\dagger t\))

\(n = \min(N_L, N_R)\)
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\(n = \min(N_L, N_R)\)

RMT approach: random unitary S-matrices with uniform distribution

\(S^T = S\) (+symmetric)

\(S^\dagger = S^{-1}\)

\(S^R = S\) (+self-dual)

\(\beta = 1\) (COE)

\(\beta = 2\) (CUE)

\(\beta = 4\) (CSE)
Possible ways to attack the problem

\[ \langle p \rangle = \langle \text{tr} [tt^\dagger(1 - tt^\dagger)] \rangle \]

\[ \quad \leftrightarrow \text{integration over unitary group:} \]
\[ \quad \text{doable at } N_{L,R} \gg 1 \text{ or } N_{L,R} = 1 \]
\[ \quad \text{hard at arbitrary } N_{L,R} \]
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\[ \langle p \rangle = \int_0^1 dT \, \rho(T) \, T(1 - T) \]

\[ \leftrightarrow \text{linear statistic on } \{T_p\} \text{ with mean density } \rho(T) = \sum_p \langle \delta(T - T_p) \rangle : \]
\[ \rho(T) \propto 1/\sqrt{T(1 - T)} , \quad N_{L,R} \gg 1 \]
\[ \rho(T) \propto T^{\beta/2 - 1} , \quad N_{L,R} = 1 \]

no analytical results at arbitrary \( N_{L,R} \)
Joint PDF of transmission eigenvalues $T_p$ is well known: Beenakker: ’97

$$\mathcal{P}(\{T_i\}) = N_\beta^{-1} \prod_{j<k} |T_j - T_k|^{\beta} \prod_{i=1}^{n} T_i^{\alpha-1}, \quad \alpha = \frac{\beta}{2}(|N_L - N_R| + 1)$$
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Key idea — relation of $P_\beta(\{T_p\})$ to Selberg’s integral:

$$\int_{0}^{1} \cdots \int_{0}^{1} |\Delta(T)|^{2c} \prod_{j=1}^{n} T_j^{a-1} (1 - T_j)^{b-1} dT_j = \prod_{j=0}^{n-1} \frac{\Gamma(1+c+jc)\Gamma(a+jc)\Gamma(b+jc)}{\Gamma(1+c)\Gamma(a+b+(n+j-1)c)}$$

as integral kernel of SI at $a = \alpha$, $b = 1$ and $c = \frac{\beta}{2}$

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Beenakker: ‘97

Savin & Sommers: ‘06
Statistics and Selberg’s integral

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  \[\text{Savin & Sommers: '06}\]

- SI as a multidimensional generalization of Euler’s beta-function \[\rightsquigarrow\] recursions from $\langle T_1 \cdots T^k_m \rangle = \langle T_1 \cdots \frac{\partial}{\partial T_m} \frac{T_{m}^{k+1}}{k+1} \rangle$ and partial integration
Statistics and Selberg’s integral

- Joint PDF of transmission eigenvalues $T_p$ is well known:  
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  recursions from $\langle T_1 \cdots T_k \rangle = \langle T_1 \cdots \frac{\partial}{\partial T_m} \frac{T_m^{k+1}}{k+1} \rangle$ and partial integration

- Method is applicable for both linear and nonlinear statistics:
  \[ \triangleright \] set of algebraic relations for $\langle T_1^{n_1} \cdots T_k^{n_k} \rangle$ derived in closed form
  \[ \triangleright \] explicit realisation done for $\sum_i n_i \leq 4$
The following two relations do all the job in this case:

\[ \langle T_1^2 \rangle = \frac{[\alpha + 1 + \beta(n - 1)] \langle T_1 \rangle - \frac{\beta(n - 1)}{2} \langle T_1 T_2 \rangle}{\alpha + 2 + \beta(n - 1)} , \quad \langle T_1 \cdots T_m \rangle = \prod_{j=1}^{m} \frac{\alpha + \frac{\beta}{2}(n - j)}{\alpha + 1 + \frac{\beta}{2}(2n - j - 1)} \]
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\]

Average: \[ \langle g \rangle \equiv n\langle T_1 \rangle = \frac{N_L N_R}{N-1+\frac{2}{\beta}}, \quad N = N_L + N_R \]

Variance: \[ \text{var}(g) \equiv n\langle T_1^2 \rangle + n(n-1)\langle T_1 T_2 \rangle - n^2\langle T_1 \rangle^2 \]

\[
= \frac{2N_L(N_L-1+\frac{2}{\beta})N_R(N_R-1+\frac{2}{\beta})}{\beta(N-2+\frac{2}{\beta})(N-1+\frac{2}{\beta})^2(N-1+\frac{4}{\beta})}
\]

Shot noise: \[ \langle p \rangle \equiv n[\langle T_1 \rangle - \langle T_1^2 \rangle] = \frac{N_L(N_L-1+\frac{2}{\beta})N_R(N_R-1+\frac{2}{\beta})}{(N-2+\frac{2}{\beta})(N-1+\frac{2}{\beta})(N-1+\frac{4}{\beta})} \]
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\( \langle p \rangle \) also studied independently by means of

- semiclassical trajectory approach at \( \beta = 1, 2 \) \hspace{1cm} \text{Braun et al.: '06}
- RMT calculation at \( N_L = N_R \) and \( \beta = 1, 2 \) \hspace{1cm} \text{Bulgakov et al.: '06}
Fano factor

- Relationship between $\langle g \rangle$, $\text{var}(g)$ and $\langle p \rangle$: 
  \[
  \frac{2 \langle p \rangle \langle g \rangle}{\beta \text{var}(g)} = N_L N_R
  \]

- Fano factor (noise suppression w.r.t. uncorrelated transport):
  \[
  F = \frac{\langle p \rangle}{p_{\text{pois}}} = \frac{(N_L - 1 + \frac{2}{\beta})(N_R - 1 + \frac{2}{\beta})}{(N - 2 + \frac{2}{\beta})(N - 1 + \frac{4}{\beta})}
  \]

Symmetric cavities, $N_{L,R} = n$:

Asymmetric cavities, $N_L \neq N_R$: 

![Graph showing Fano factor for symmetric and asymmetric cavities](image-url)
Transmitted charge

...is a linear statistic (fully characterised by \( \rho(T) \))

- Cumulants \( \langle \langle Q^m \rangle \rangle \) of the transmitted charge and moments \( \langle T^m \rangle \):
  \[
  \sum_i \langle \ln[1 + T_i (e^\lambda - 1)] \rangle = \sum_m \frac{\lambda^m}{m!} \langle \langle Q^m \rangle \rangle
  \]
  (Levitov & Lesovik: '93)

  The first 2 cumulants are the average conductance and shot-noise
Transmitted charge

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- 3rd cumulant (vanishes at $\beta = 2$ and $N_L = N_R$):

$$\langle\langle Q^3 \rangle\rangle = \langle\langle Q^2 \rangle\rangle \frac{[(1 - \frac{2}{\beta})^2 - (N_L - N_R)^2]}{(N - 3 + \frac{2}{\beta})(N - 1 + \frac{6}{\beta})}$$

Levitov & Lesovik: '93

Novaes: '07
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  Novaes: '07

- 4th cumulant is obtained at arbitrary $N_{L,R}$. Limiting cases:
  \[ \langle\langle Q^4 \rangle\rangle = -\frac{2}{105}, -\frac{1}{30}, -\frac{1}{30} \text{ at } \beta = 1, 2, 4 \text{ and } N_{L,R} = 1 \]
  \[ \text{large } N \text{ expansion reproduced} \]
  Blanter, Schomerus, Beenakker: '01
  \[ \frac{\langle\langle Q^4 \rangle\rangle}{\langle\langle Q^2 \rangle\rangle} = \frac{1}{N^4} \left[ N_L^4 - 4N_LN_R(2N_L^2 - 3N_LN_R + 2N_R^2) + N_R^4 + \frac{6(\beta - 2)}{\beta N} (N_L - N_R)^2 (2N_L^2 - 7N_LN_R + 2N_R^2) \right] + O\left(\frac{1}{N^6}\right) \]
are examples of nonlinear statistics (characterised by different $T_p$)

- Exact RMT result for the conductance skewness:

$$
\langle \langle g^3 \rangle \rangle = \text{var}(g) \frac{4[(1 - \frac{2}{\beta})^2 - (N_L - N_R)^2]}{\beta(N - 3 + \frac{2}{\beta})(N - 1 + \frac{2}{\beta})(N - 1 + \frac{6}{\beta})}
$$

It vanishes for $N_L = N_R$ at $\beta = 2$ as any odd cumulant in this case. Distribution of $g$ becomes symmetric around $\frac{n}{2}$ in this case.
Skewness & kurtosis of conductance

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- Kurtosis of $g$ is obtained explicitly at arbitrary $N_{L,R}$. Limiting cases:
  $\triangleright$ $\langle\langle g^4 \rangle\rangle = -\frac{32}{4725}, -\frac{1}{120}, -\frac{1}{540}$ at $\beta = 1, 2, 4$ and $N_{L,R} = 1$

  $\triangleright$ large $N$ expansion
  $$\langle\langle g^4 \rangle\rangle \frac{1}{\text{var}(g)} = \frac{24}{\beta^2 N^6} \left[ (N_L - N_R)^2 (N_L^2 + N_R^2 - 4N_L N_R) \right. \\
  \left. + \frac{\beta - 2}{\beta N} (12(N_L^4 + N_R^4) - 64N_L N_R(N_L^2 + N_R^2) + 105N_L^2 N_R^2) \right] + O\left(\frac{1}{N^4}\right)$$

- Conductance distribution gets more Gaussian-like as $N$ grows.
Variance of shot-noise

Variance of $p$ is obtained at arbitrary $N_{L,R}$. Limiting cases:

\[ \text{var}(p) = \frac{4}{525}, \frac{1}{180}, \frac{1}{180} \text{ at } \beta = 1, 2, 4 \text{ and } N_{L,R} = 1 \]

\[ \text{var}(p) = \frac{2}{\beta N_5} \left[ N_L^4 + N_R^4 - 4N_LN_R(N_L - N_R)^2 \right. \\
\left. + \frac{\beta-2}{\beta N} (9(N_L^4 + N_R^4) - 42N_LN_R(N_L^2 + N_R^2) + 70N_L^2N_R^2) \right] + O\left(\frac{1}{N_3}\right) \]

Symmetric cavities, $N_{L,R} = n$:

Asymmetric cavities, $N_L \neq N_R$:

\[ \text{var}(p) \approx \frac{1}{64\beta} \left( 1 + \frac{\beta-2}{\beta n} \right) \text{ sets up universal Gaussian fluctuations at } N \gg 1 \]
... have finite support: $0 < g < n \equiv L_g$ and $0 < p < \frac{n}{4} \equiv L_p$
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\[ \sim \] can be represented as Fourier series

\[ P_x^{(\beta)}(x) = \sum_{m=1}^{\infty} \frac{2}{L_x} \sin \left( \frac{m \pi x}{L_x} \right) C_x^{(\beta)}(m), \quad \text{with} \quad x = g, p \]

with $C_x^{(\beta)}(m)$ being exactly determined at $\beta = 1, 2, 4$. Examples:

\[
\begin{align*}
C_g^{(2)}(m) &= n! N_{\beta=2}^{-1} \Im \text{det} \left[ \int_0^1 dT \, T^{\alpha+l+k-3} e^{im\pi T/n} \right] \\
C_g^{(4)}(m) &= n! N_{\beta=4}^{-1} \Im \text{Pfaff} \left[ (l - k) \int_0^1 dT \, T^{\alpha+k+l-4} e^{im\pi T/n} \right]
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\[ C_g^{(4)}(m) = n! N_{\beta=4}^{-1} \Im Pfaff \left[ (l-k) \int_0^1 dT T^{\alpha+k+l-4} e^{im\pi T/n} \right] \]

- Nonanalyticity (weak) at $x = \frac{k}{n} L_x$, $k = 1, \ldots, n$, its geometrical meaning: integration over plane (sphere) cut by $n$-cube
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- Nonanalyticity (weak) at $x = \frac{k}{n} L_x$, $k = 1, \ldots, n$, its geometrical meaning: integration over plane (sphere) cut by $n$-cube
- Gaussian-like behaviour at $N_{L,R} \gg 1$ in the bulk ($x \sim L_x/2$)
- Power-law dependence at the edges of support ($x \sim 0$ or $x \sim L_x$)

\(~\sim\) “two phase transition” in Coulomb gas problem

Vivo, Majumdar, Bohigas: arXiv:0809.0999
... can be exactly determined up to linear order in distances from edges:

\[ P_x(x) \simeq A_x x^{\ell_x} \] (left) \quad \text{and} \quad \[ P_x(x) \simeq B_x (L_x - x)^{r_x} \] (right)

E.g., conductance: \( \ell_g = n[\alpha + \frac{\beta}{2}(n - 1)] - 1 \) and \( r_g = (n - 1)(1 + \frac{\beta}{2}n) \)

All constants can be reduced to some forms of Selberg’s integral (any \( \beta \))

Conductance \((\beta=1, N_L=4, N_R=2)\):

Shot-noise (the same parameters):
Conclusions

- **Relation** of Selberg’s integral to quantum transport in chaotic cavities
- **Method** developed to find recursion for moments \( \langle T_{i_1}^{n_1} \cdots T_{i_k}^{n_k} \rangle \)
- **Explicit** implementation for \( \sum_i n_i \leq 4 \) yields **exact results** for:
  - First four cumulants of transmitted charge & conductance
  - Average & variance of the shot-noise power
- **Experimentally accessible predictions**
- **Distributions** found in terms of Fourier series expansion
  - **Asymptotics** determined exactly near the edges of support
Conclusions

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  - Average & variance of the shot-noise power
- **Experimentally accessible predictions**
- **Distributions** found in terms of Fourier series expansion
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  ($\beta = 2$ already done Novaes: ’08, Osipov, Kanzieper: ’08)
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