On extension of the formalism MPDFA and its application to the analyses of DNS 4096 conducted by Kaneda and Ishihara

T. Arimitsu (U of Tsukuba)
N. Arimitsu (Yokohama Nat’l U)
Multi-fractal PDF Theory (MPDFT)

is

a statistical mechanical ensemble theory for analyzing those phenomena providing fat-tail PDFs, constructed by authors under the assumption that the singularities due to the scale invariance of the Navier-Stokes equation for high Reynolds number distribute themselves multi-fractal way in real physical space.

PDF: Probability Density Function
Contents of the talk

In this talk, we will analyze, first, the PDFs of energy transfer rates, extracted by Kaneda and Ishihara from their $4096^3$DNS, in order to show how MPDFT works in the precise analyses of the PDFs, and how it can provide us with new information to understand turbulence (Details are explained at the poster presentation by Prof. N. Arimitsu).

Then, we will show that the formalism MPDFT itself is telling us that the origin of the multi-fractal character of the fully developed turbulence is deeply related to the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits, where $\delta$ is the ratio of distances of two measuring points or diameters of measuring areas (Details are given at the poster presentation by Prof. Motoike).
Intermittency in terms of MPDFT

- **MPDFT** starts with the **scale invariance** of the incompressible Navier-Stokes equation for high Reynolds number.
- “**Singularities**”, due to the invariance, appear in **velocity derivatives**, **energy transfer rates** and so on, whose **degrees of singularity** are specified by an exponent $\alpha$.
- The **singularities** specified by $\alpha$ are assumed to distribute themselves in physical space with a **fractal dimension** $f(\alpha)$.
- The **probability** $P^{(n)}(\alpha)\ d\alpha$, to find a singularity within the range $\alpha \sim \alpha + d\alpha$ at a point in physical space in the $n$th multi-fractal depth, is assumed to be given by the Tsallis-type distribution function. Whereas, $n_\delta = n/\ln \delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits.
- Within the formalism **MPDFT**, $\delta$ is the **ratio of distances** of two measuring points or of diameters of measuring areas, which provides us with a new interpretation that **fully developed turbulence** is an accumulation of $\delta$-scale Cantor sets.
Intermittency in terms of MPDFT

MPDFT starts with the scale invariance of the incompressible Navier-Stokes equation for high Reynolds number.

- The probability $P^{(n)}(\alpha) d\alpha$, to find a singularity within the range $\alpha \sim \alpha + d\alpha$ at a point in physical space in the $n_\delta$-th multi-fractal depth, is assumed to be given by the Tsallis-type distribution function, where $n_\delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^{\infty}$-periodic orbits.
- Within the formalism MPDFT, $\delta$ is the ratio of distances of two measuring points or of diameters of measuring areas, which provides us with a new interpretation that fully developed turbulence is an accumulation of $\delta$-scale Cantor sets.
Navier-Stokes equation for an incompressible fluid

\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} \left( \frac{p}{\rho} \right) + \nu \nabla^2 \vec{u} \]

\[ \rho \text{: the mass density} \]
\[ p \text{: the pressure} \]
\[ \nu \text{: the kinematic viscosity} \]

For high Raynolds number \( \text{Re} = \frac{\delta u_0 \ell_0}{\nu} \gg 1 \),

it is invariant under the scale transformation \((\alpha : \text{real})\)

\[ \vec{r} \rightarrow \lambda \vec{r}, \quad \vec{u} \rightarrow \lambda^{\alpha/3} \vec{u}, \quad t \rightarrow \lambda^{1-\alpha/3} t, \quad \frac{p}{\rho} \rightarrow \lambda^{2\alpha/3} \frac{p}{\rho}. \]
Intermittency in terms of MPDFT

• MPDFT starts with the scale invariance of the incompressible incompressible Navier-Stokes equation for high Reynolds number.

“Singularities”, due to the invariance, appear in velocity derivatives, energy transfer rates and so on, whose degrees of singularity are specified by an exponent $\alpha$.

distribution function, where $n_\delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits.

• Within the formalism MPDFT, $\delta$ is the ratio of distances of two measuring points or of diameters of measuring areas, which provides us with a new interpretation that fully developed turbulence is an accumulation of $\delta$-scale Cantor sets.
“Singularities”

With the length \( \ell_n = \delta_n \ell_0 \) \((\delta_n = \delta^{-n})\), these quantities are given as follows.

\[
|\vec{u}'| = \lim_{n \to \infty} u'_n = \lim_{\ell_n \to 0} \frac{\delta u_n}{\ell_n} \sim \lim_{\ell_n \to 0} \ell_n^{\frac{1}{3} \alpha - 1}, \quad \delta u_n = |u(\bullet + \ell_n) - u(\bullet)|,
\]

\[
|\vec{a}| = \lim_{n \to \infty} a_n = \lim_{\ell_n \to 0} \frac{\delta p_n}{\ell_n} \sim \lim_{\ell_n \to 0} \ell_n^{\frac{2}{3} \alpha - 1}, \quad \delta p_n = \left(\frac{p}{\rho}\right)(\bullet + \ell_n) - \left(\frac{p}{\rho}\right)(\bullet),
\]

\[
\varepsilon_\infty = \lim_{n \to \infty} \varepsilon_n = \lim_{\ell_n \to 0} \left(\frac{\ell_n}{\ell_0}\right)^{\alpha - 1} \sim \lim_{\ell_n \to 0} \ell_n^{\alpha - 1}
\]

where \( \vec{a} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \) \(\vec{u}\) is the acceleration of fluid particle.

The velocity derivative, the acceleration and the energy transfer rate become, respectively, singular for \(\alpha < 3\), \(\alpha < 1.5\) and \(\alpha < 1\) in the limit \(\ell_n \to 0\).

When \(\alpha < 1\), all these three quantities become large for each \(\ell_n\).
Intermittency in terms of MPDFT

- **MPDFT** starts with the scale invariance of the incompressible Navier-Stokes equation for high Reynolds number.
- “Singularities”, due to the invariance, appear in velocity derivatives, energy transfer rates, and so on, whose degrees of il ith it ifi id b t

The **singularities** specified by $\alpha$ are assumed to distribute themselves in physical space with a **fractal dimension** $f(\alpha)$.

- In the range $\alpha \sim \alpha + d\alpha$ at a point in physical space in the $n_\delta$-th multi-fractal depth, is assumed to be given by the Tsallis-type distribution function, where $n_\delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits.
- Within the formalism **MPDFT**, $\delta$ is the ratio of distances of two measuring points or of diameters of measuring areas, which provides us with a new interpretation that fully developed turbulence is an accumulation of $\delta$-scale Cantor sets.
Eddies with size $\ell_n$ painted by the same color carry singularity labeled by $\alpha$, and occupy real space with the fractal dimension $f(\alpha)$. 

Origin of intermittency
Intermittency in terms of MPDFT

- **MPDFT** starts with the scale invariance of the incompressible Navier-Stokes equation for high Reynolds number.
- “Singularities”, due to the invariance, appear in velocity derivatives, energy transfer rates and so on, whose degrees of ill-posedness are specified by an exponent $\alpha$.

The probability $P^{(n)}(\alpha)\,d\alpha$, to find a singularity within the range $\alpha \sim \alpha + d\alpha$ at a point in physical space in the $n$th multi-fractal depth, is assumed to be given by the Tsallis-type distribution function. Whereas, $n_\delta = n/\ln \delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits.

This provides us with a new interpretation that fully developed turbulence is an accumulation of $\delta$-scale Cantor sets.
A&A model assumes that the distribution of $\alpha$ is given by the Tsallis-type distribution function.

\[ P^{(n)}(\alpha) = \frac{1}{Z^{(n)}} \left( 1 - \frac{(\alpha-\alpha_0)^2}{(\Delta \alpha)^2} \right)^{n_\delta/(1-q)} = \frac{1}{Z^{(n)}} \left( 1 - \frac{(\alpha-\alpha_0)^2}{(\Delta \alpha)^2} \right)^{n/(1-q)\ln \delta} \]

\[ (\Delta \alpha)^2 = \frac{2X}{(1-q)\ln \delta}, \quad n = n_\delta \ln \delta \]

\[ f(\alpha) = 1 - \frac{1}{(1-q)\ln \delta} \ln \left( 1 - \frac{(\alpha-\alpha_0)^2}{(\Delta \alpha)^2} \right) \]

\[ \langle \varepsilon_n / \varepsilon \rangle = 1, \quad \langle (\varepsilon_n / \varepsilon)^2 \rangle = \delta_n^{-\mu} \]

Three parameters $\alpha_0, X, q$ are determined by the conditions

\[ \ln 2 \left( \frac{1}{(1-q)\ln \delta} \right) = \left( \frac{1}{\alpha_-} - \frac{1}{\alpha_+} \right) \]

Note that $\langle \cdots \rangle$ is taken with $P^{(n)}(\alpha)$.

$\alpha_0 = \alpha_0(\mu), \quad X = X(\mu), \quad q = q(\mu)$

\[ f(\alpha_\pm) = 0 \]
Relation of the present approach to others.

Log-normal model (K62)

\[
P^{(n)}(\alpha) = \sqrt{\frac{n}{2\pi\sigma^2}} \exp \left[ -\frac{n(\alpha - \alpha_0)^2}{2\sigma^2} \right]
\]

Two parameters \( \alpha_0, \sigma \) are determined by the conditions

\[
\langle \varepsilon_n / \varepsilon \rangle = 1, \quad \langle (\varepsilon_n / \varepsilon)^2 \rangle = \delta_n^{(-\mu)}.
\]

Note that \( \langle \cdots \rangle \) is taken with \( P^{(n)}(\alpha) \).

\[
\alpha_0 = \alpha_0(\mu), \quad \sigma = \sigma(\mu)
\]
Comments (2/2)

\[ p = \frac{1 + \sqrt{2^\mu - 1}}{2}. \]

*p* model (Meneveau-Sreenivasan 1987)

\[
P^{(n)}(\alpha) \propto \frac{1}{2^y y^y (1 - y)^{1-y}} n, \quad y = \frac{\alpha + \log_2(1 - p)}{\log_2[(1 - p)/p]}
\]

The parameter \( p \) are determined by the conditions

\[
\langle (\varepsilon_n / \varepsilon)^2 \rangle = \delta_n^{(-\mu)}.
\]

Note that \( \langle \cdots \rangle \) is taken with \( P^{(n)}(\alpha) \).

Note that \( p \) model preserves energy transfer rates from the beginning by its own characteristics.
Scaling exponents of velocity structure function

Competition in $\mathcal{S}_m$
Scaling exponents $\zeta_m$ of velocity structure function

\[ \langle (\delta u_n)^m \rangle \propto \ell_n^{\zeta_m} \]

with $\delta u_n = |u(\bullet + \ell_n) - u(\bullet)|$,

\[ \langle \cdots \rangle = \int d\alpha \cdots P(\alpha). \]

**K41 (1941)**
\[ \zeta_m = m/3 \]

**Log-normal (1962)**
\[ \zeta_m = m/3 - \mu (m-3)/18 \]

**$\beta$-model (1978)**
\[ \zeta_m = m/3 - \mu (m-3)/3 \]

**p-model (1987)**
\[ \zeta_m = 1 - \log_2 [p^{m/3} + (1-p)^{m/3}] \]
\[ p = \frac{1 + (2\mu - 1)^{1/2}}{2} \]

**Log-Poisson (1994)**
\[ \zeta_m = m/9 + 2(1 - (2/3)^{m/3}) \]

**Present model (2000)**
\[ \zeta_m = \frac{\alpha_0 m}{3} - \frac{2X m^2}{9(1+\sqrt{C_{m/3}})} - \frac{1}{1-q} \left[ 1 - \log_2 \left( 1 + \sqrt{C_{m/3}} \right) \right] \]

with
\[ C_q = 1 + 2X \bar{q}^2 (1-q) \ln 2 \]
\[ \sqrt{2X} = \frac{\sqrt{\alpha_0^2 + (1-q)^2} - (1-q)}{\sqrt{b}}, \quad b = \frac{1 - 2^q - 1}{(1-q) \ln 2}. \]
As for the **new** scaling relation

\[
\frac{\ln 2}{(1-q) \ln \delta} = \left( \frac{1}{\alpha_-} - \frac{1}{\alpha_+} \right)
\]
Let us look at the $3^n$ periodic orbits in the bifurcation diagram of Logistic Map.

This corresponds to the case where $\delta = 3$. 
Bifurcation Diagram of Logistic Map

window of 3 period
Bifurcation Diagram of Logistic Map

window of $3^2$ period
Bifurcation Diagram of Logistic Map

window of $3^3$ period
$\delta$-scale Cantor sets for the $\delta^\infty$ super-stable orbit

$\begin{align*}
n_2 &= 0 \\
n_2 &= 1 \\
n_2 &= 2
\end{align*}$

$2^\infty$ super-stable orbits

Structure of stages for $2^\infty$ super-stable orbits

$\begin{align*}
n_3 &= 0 \\
n_3 &= 1 \\
n_3 &= 2
\end{align*}$

$3^\infty$ super-stable orbits

Structure of stages for $3^\infty$ super-stable orbits

30 September 2008 Newton Institute, Cambridge
Orbital expansion rates for $3^{12}$ periodic orbits

New scaling relation picks up the most intermittent evolutions out of the time series for $\delta^\infty$ periodic orbits.

→ Details are given at the poster presentation by Dr. Motoike.

30 September 2008 Newton Institute, Cambridge
Intermittency in terms of MPDFT

- MPDFT starts with the scale invariance of the incompressible Navier-Stokes equation for high Reynolds number.
- “Singularities”, due to the invariance, appear in velocity derivatives, energy transfer rates and so on, whose degrees of singularity are specified by an exponent $\alpha$.
- The singularities specified by $\alpha$ are assumed to distribute themselves in physical space with a fractal dimension $f(\alpha)$.

Within the formalism MPDFT, $\delta$ is the ratio of distances of two measuring points or of diameters of measuring areas, which provides us with a new interpretation that fully developed turbulence is an accumulation of $\delta$-scale Cantor sets.
accumulation of $\delta$-scale Cantor sets

A sketch of turbulence by Leonardo da Vinci
The PDF \( \hat{\Pi}^{(n)}(\xi_n)d\xi_n = \Pi^{(n)}(x_n)dx_n \) for quantity \( x_n \) with the normalized variable \( \xi_n = x_n/\sqrt{\langle x_n^2 \rangle} \) is given by

\[
\hat{\Pi}^{(n)}(\xi_n) = \prod^{(n)}_{\phi} \left\{ 1 - \frac{1-(q)^{1+3f(\alpha^*)/\phi}}{2} \left[ (\xi_n/\xi_n^*)^w - 1 \right] \right\}^{1/(1-q)} \quad \text{for } |\xi_n| \leq \xi_n^* \quad (\alpha \geq \alpha^*)
\]

\[
\hat{\Pi}^{(n)}(\xi_n) = \prod^{(n)}_{\phi} \left[ 1 - \frac{1-q}{n_\delta} \left( \frac{3\ln|\xi_n/\xi_{n,0}|}{2\phi^2 X|\ln\delta_n|} \right)^2 \right]^{n_\delta/(1-q)} \quad \text{for } \xi_n^* \leq |\xi_n| \quad (\alpha^* \geq \alpha)
\]

with \( |\xi_{n,0}| = \xi_n^* \delta_n^{2\alpha_0/3-3\phi/2} \).

- \( \phi = 1 \): velocity fluctuations and derivatives
- \( \phi = 2 \): fluid particle accelerations
- \( \phi = 3 \): energy transfer rates

\[
\xi_n = \xi_n^* \delta_n^{2\alpha/3-3\phi/2}
\]

\( \zeta_m \) is the scaling exponent of velocity structure function.

The connection points are around \( \xi_n^* \approx 0.5 \sim 1.4 \).

Here, \( \alpha^* \approx 0.9 \sim 1.1 \) is determined by analyzing data.

Remember that variables become singular for \( \alpha < 1 \).
It is revealed in the analyses of experimental data that there are two mechanisms contributing to the PDFs, i.e.,

- one is for the **tail part**, and
- the other for the **center part**.
PDFs within A&A model

\[ \ell_n = \delta_n \ell_0 \quad (\delta_n = 2^{-n}) \]
Competition in PDF


- $1024^3$DNS by Gotoh 2000, 2002
- Experiment by Bodenschatz 2001
Fig. 6.6. Analyses of the PDF’s of fluid particle accelerations, measured by Gotoh et al. at $R_\lambda = 380$ (circles in the top set) and by Bodenschatz et al. at $R_\lambda = 690$ (circled in the bottom set), by means of the PDF’s $\langle \omega_n \rangle$ by the harmonious representation (solid line) and by the log-normal model (dashed line) are plotted on (a) log and (b) linear scales. The PDF’s by the $p$ model (dotted line) are compared with the PDF’s by the harmonious representation (solid line). Results are displayed in pairs. The solid lines in each set of pairs are the same. For better visibility, each PDF is shifted by $-2$ unit in (a) and by $-0.4$ in (b) along the vertical axis. Parameters are given in the text.
Competition in PDFs of velocity fluctuations (Gotoh 2001)

Fig. 6.5. Analyses of the PDF’s of the velocity fluctuations (closed circles) for three different measuring distances, observed by Gotoh et al. at \( R_\lambda = 380 \), with the help of the PDF’s \( \tilde{H}^{(n)}(\xi_n) \) by the harmonious representation (solid line) and by the log-normal model (dashed line) are plotted on (a) log and (b) linear scales. The PDF’s by the \( p \) model (dotted line) are compared with the PDF’s by the harmonious representation (solid line). Comparisons are displayed in pairs. The solid lines in each set of pairs are the same. For better visibility, each PDF is shifted by \(-2\) unit in (a) and by \(-0.2\) in (b) along the vertical axis. Parameters are given in the text.
Analysis
of
PDF for energy transfer rates
extracted by
Kaneda-Ishihara’s Group
from their $4096^3$DNS
in terms of
MPDFT
PDF of energy transfer rates  (DNS)

4096³ DNS PDF by Kaneda-Ishihara group

2r/η from top to bottom:

5.79, 6.88, 8.18, 9.73, 11.6, 13.8, 16.4, 19.5, 23.1, 27.5, 32.8, 38.9, 46.3, 55.1, 65.6, 78.0, 92.7, 110, 131, 156, 186, 221, 264, 314, 374, 442, 532, 628, 748, 898, 1083, 1257, 1496, 1848, 2244, 2618,

For better visibility, each PDF is shifted by −1 unit along the vertical axis.
PDF of energy transfer rates \( (\delta = 2^{1/4}) \)

Delta = 2^{1/4}

Closed circles
DNS PDF by Kaneda & Ishihara group

Lines
Theoretical PDF with
\[
(1-q) \ln \delta = 0.323, \\
\alpha_0 = 1.19, \\
\chi = 0.382 \\
(\mu = 0.320)
\]

\[ q = -0.864 \]

For better visibility, each PDF is shifted by –1 unit along the vertical axis.
PDF of energy transfer rates ($\delta = 2^{1/4}$)

For better visibility, each PDF is shifted by –1 unit along the vertical axis.

$\xi_n$: normalized by its deviation

$\delta$: number of multifractal steps

$\xi_n$: normalized by its deviation

$q = -0.864$

$X = 0.382$

& Ishihara

$\mu = 0.320$

$\alpha_0 = 1.19$

$X = 0.382$

Fractal steps

s deviation

is shifted by –1

30 September 2008
Newton Institute, Cambridge
PDF of energy transfer rates \((\delta = 2^{1/4})\)

delta = 2^{1/4}

Closed circles
DNS PDF by Kaneda & Ishihara group

Lines
Theoretical PDF with
\[(1-q) \ln \delta = 0.323,\]
\[\alpha_0 = 1.19,\]
\[\chi = 0.382\]
\[(\mu = 0.320)\]

For better visibility, each PDF is shifted by –0.1 unit along the vertical axis.
PDF of energy transfer rates ($\delta = 2^{1/2}$)

Closed circles
DNS PDF by Kaneda & Ishihara group

Lines
Theoretical PDF with

$$(1-q) \ln \delta = 0.323,$$

$\alpha_0 = 1.19,$
$X = 0.382$
$(\mu = 0.320)\]

$q = 0.068$

For better visibility, each PDF is shifted by –1 unit along the vertical axis.
PDF of energy transfer rates \((\delta = 2^{1/2})\)

\[
\Pi(n) < \Pi(n) < \Pi(n) < \Pi(n)
\]

Closed circles
DNS PDF by Kaneda & Ishihara group

Lines
Theoretical PDF with
\[(1-q) \ln \delta = 0.323, \quad \alpha_0 = 1.19, \quad \chi = 0.382 \quad (\mu = 0.320)\]

\[q = 0.068\]

For better visibility, each PDF is shifted by –0.1 unit along the vertical axis.
PDF of energy transfer rates \((\delta = 2)\)

Closed circles
DNS PDF by Kaneda & Ishihara group

Lines
Theoretical PDF with
\[(1-q) \ln \delta = 0.323,\]
\[\alpha_0 = 1.19,\]
\[X = 0.382\]
\[(\mu = 0.320)\]

\[q = 0.534\]

For better visibility, each PDF is shifted by –1 unit along the vertical axis.
**PDF of energy transfer rates** \((\delta = 2)\)

### Closed circles
DNS PDF by Kaneda & Ishihara group

### Lines
Theoretical PDF with
\[
(1-q) \ln \delta = 0.323, \quad \alpha_0 = 1.19, \quad X = 0.382, \quad (\mu = 0.320)
\]

\(q = 0.534\)

For better visibility, each PDF is shifted by \(-0.1\) unit along the vertical axis.
The dependence of $n_\delta$ on $2r/\eta$

for energy transfer rates

$$n_\delta = -\log_\delta \left( \frac{\ell_n}{\eta} \right) + n_0\delta$$

$\delta = 2$,  
$$n_\delta = -1.01 \times \log_\delta \left( \frac{2r}{\eta} \right) + 9.76$$

$\delta = 2^{1/2}$,  
$$n_\delta = -1.02 \times \log_\delta \left( \frac{2r}{\eta} \right) + 22.8$$

$\delta = 2^{1/4}$,  
$$n_\delta = -1.02 \times \log_\delta \left( \frac{2r}{\eta} \right) + 48.6$$

$n_\delta$ is the number of stages in the $\delta$-scale Cantor sets created from $\delta^\infty$-periodic orbits.

→Details are given at the poster presentation by Dr. N. Arimitsu.

Self-similarity in the inertial sub-range.

30 September 2008  
Newton Institute, Cambridge
The dependence of $n$ on $2r/\eta$ for energy transfer rates

\[ n = n_\delta \ln \delta \]

\[ n = -\ln \left( \frac{\ell_n}{\eta} \right) + \ln \left( \frac{\ell_0}{\eta} \right) \]

$\delta = 2$, \quad $n = -1.01 \times \ln \left( \frac{2r}{\eta} \right) + 9.76$

$\delta = 2^{\frac{1}{2}}$, \quad $n = -1.02 \times \ln \left( \frac{2r}{\eta} \right) + 9.74$

$\delta = 2^{\frac{1}{4}}$, \quad $n = -1.02 \times \ln \left( \frac{2r}{\eta} \right) + 9.71$

Self-similarity in the inertial sub-range.

\[ \frac{\ell_0}{3\eta} \approx 5,470 \sim 5,770 \]

\[ \approx \frac{3}{2} \times 4,096 \]

Details are given at the poster presentation by Dr. N. Arimitsu.
The dependence of $\xi_n^*$ on $2r/\eta$ for energy transfer rates

Self-similarity in the inertial sub-range.

- $\delta = 2$, $\xi_n^* = 0.0790 \times \ln\left(\frac{2r}{\eta}\right) + 0.185$
- $\delta = 2^{1/2}$, $\xi_n^* = 0.0814 \times \ln\left(\frac{2r}{\eta}\right) + 0.132$
- $\delta = 2^{1/4}$, $\xi_n^* = 0.0822 \times \ln\left(\frac{2r}{\eta}\right) + 0.114$

Details are given at the poster presentation by Dr. N. Arimitsu.
• **MPDFT** (Multi-fractal PDF Theory) explains quite well the fat-tail PDFs obtained by numerical and ordinary experiments.

• Self-similarity (intemittency) is conspicuous even in the sub-inertial range, since $n_\delta$ changes its values almost by 1 (Note that $n_\delta$ is representing the number of stages in $\delta$-scale Cantor sets, therefore, should be dependent on $\delta$).

• The multi-fractal depth $n$ ($= n_\delta \ln \delta$) corresponds, one to one, to the distance $2r$ of two measuring points, or of the diameter of an eddy. It is remarkable that $n$ does not dependent on $\delta$. 
Summary (2/2)

• The largest size $\ell_0$ within MPDFT is estimated to be about 3 times larger than the possible largest eddy size in the DNS by Kaneda-Ishihara. It is about 8 times larger than the integral scale $L$. It may suggest that, for the precise interpretation of $\ell_0$, we need some kind of “renormalization” procedure.

• The connection points $\xi_n*$ between the center part and the tail part of PDFs also represent the self-similarity, i.e., $\xi_n*$ increases in proportion to the increase of $\ln(2r)$.

• Important information for the center part of PDF is obtained through the analyses within MPDFT.
Thank you for your attention.