Resolving the cascade bottleneck in vortex-line turbulence
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Abstract

Both in many superfluid experimental situations and simulations of a 3D hard-core interaction model, it is found that the vortex line length in superfluid turbulence decays in a manner consistent with classical turbulence. Two decay mechanisms have been proposed, Kelvin wave emission along lines and phonon radiation at small scales. It has been suggested that both would require a Kelvin wave cascade, which theory says cannot reach the smallest scales due to a bottleneck. In this presentation we will discuss a new approach using a recent quaterionic formulation of the Euler equations, coupled with the local induction approximation. Without the extra quaterionic terms it can be shown that if there are sharp reconnections, the above scenario occurs. But with the extra terms, the direction of propagation of nonlinear waves is reversed, there is a cascade to the smallest scales that could create phonons, and the paradox can be resolved.

3D anti-parallel Euler generates vortex waves

Simulation of anti-parallel vortex collapse.
Outline:
• Anti-parallel geometry and its justification.
• 3D numerical DNS evidence for spirals.
• Question of cascade in superfluid turbulence
• A new vortex model: anti-parallel Biot-Savart + local induction approximation
• Question of torsion $\tau$ in Frenet-Serret formulation
• Could this resolve Euler collapse? Could this provide a cascade?

http://www.mae.cornell.edu/fdrl/research/wingvortex.html

• Crow instability between anti-parallel vortices. Vortices attract
• Typically seen in contrails formed by shedding of wing-tip vortices.
• Visualization is water vapor condensation in vortex cores.
• Reconnection ensues.
Diagram of the interaction of anti-parallel vortices. From an initial condition of anti-parallel vortices separated at their closest approach by $\delta$, if $\nu \neq 0$ there is reconnection that forms new vortices indicated by the dashed curves. However, if $\nu = 0$, a singularity can form when $\delta = 0$ if the vortices are pushed together by the self-induced strain indicated by $e$. 
Steps in idealized anti-parallel vortex reconnection taken from a low resolution, low Reynolds number calculation Melander and Hussain, 1989. From top to bottom, the first two frames show the anti-parallel vortex tubes being pushed together by self-interaction through the law of Biot-Savart. The third frame shows that reconnection has progressed to form two new tubes orthogonal to the original tubes. In the bottom frame the new tubes are separating.
Mathematical Issues

- Outstanding modern mathematical problem, $1 million prize:
- For 3D incompressible **Navier-Stokes** with finite energy, etc.
  - Find an example of a singularity of 3D incompressible Navier-Stokes
  - Prove that Navier-Stokes is regular.
  - Folk-belief: Navier-Stokes is regular
- Related: 3D incompressible **Euler**, what is known:
  - There is only one unquestionable analytic bound: Beale, Kato, Majda (1984):
    \[ \int \| \omega \|_\infty dt \to \infty \]
    (This has since also been proven using the more robust BMO norm)
  - Folk-belief: Unknown.
  - Constantin, Fefferman, Majda (1996) and Deng, Hou, Yu (2005) have further relations on time integrals of **curvature** and **velocity**:
    \[ \int \| \nabla \xi \|_\infty^2 dt \quad \int \sup |u|^2 dt \quad \int \| \nabla \xi \|_\infty |u| dt \]
Why is this popular again?

- Kerr, Phys. Fluids (1993) claimed to have a numerical solution that satisfied all these constraints.

- Hou, T.Y., & Li, R. get a different result than Kerr (1993) for:
  - Nearly the same initial condition.
  - Many more mesh points.
  - and a new J. Comp Phys. paper.
  - WHY?

- Taylor-Green.

Marc Brachet should be gearing up to run probably 4 times the resolution in each direction of the last Brachet calculation.


- Euler 250, Aussois, France, June 2007
3D Euler about a 2D Symmetry Plane
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Initial results from new calculations of interacting anti-parallel Euler vortices are presented with the objective of understanding the origins of singular scaling presented by Kerr (1993) and the lack thereof by Hou and Li (2006). Core profiles designed to reproduce the two results are presented, new more robust analysis is proposed, and new criteria for when calculations should be terminated are introduced and compared with classical resolution studies and spectral convergence tests. Most of the analysis is on a 512 × 128 × 2048 mesh, with new analysis on a just completed 1024 × 256 × 2048 used to confirm trends. One might hypothesize that there is a finite-time singularity with enstrophy growth like \( \Omega \sim (T_c - t)^{-\gamma} \) and vorticity growth like \( \|\omega\|_\infty \sim (T_c - t)^{-\gamma} \). The new analysis would then support \( \gamma_\Omega \approx 1/2 \) and \( \gamma > 1 \). These represent modifications of the conclusions of Kerr (1993). Issues that might arise at higher resolution are discussed.

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I Introduction

One definition of solving Euler’s three-dimension incompressible equations [1] is determining whether or not they dynamically generate a finite-time singularity if the initial conditions are smooth, in a bounded domain and have finite energy. The primary analytic constraint that must be satisfied [2] is:

\[ \int_0^T \|\omega\|_\infty \, dt \to \infty \]  

where \( \|\omega\|_\infty \) is the maximum of vorticity over all space. To date, Kerr (1993) [3] remains the only fully three-dimensional simulation of Euler’s equations with evidence for a singularity consistent with this and related constraints [4]. Growth of the enstrophy production and stretching along the vorticity, plus collapse of positions, supported this claim [3]. Additional weaker evidence related to blow-up in velocity and collapsing scaling functions was presented later [5].

There is only weak numerical evidence supporting these claims [6,7]. In a recent paper, as described in one of the invited talks of this symposium, Hou and Li (2006) [8] found evidence that the above scenario failed at late times.

This contribution will first comment on four issues raised at the symposium, then present preliminary new results. The four issues are:

- What criteria should be used to determine when numerical errors are substantial?
- What effect do the initial conditions have on singular trends? A cleaner initial condition is proposed.
- We introduce a new approach for determining whether there is singular behavior of the primary properties and the associated scaling. This is applied to both new and old data.

All calculations will be in the following domain: \( L_x \times L_y \times L_z = 4\pi \times 4\pi \times 2\pi \) with free-slip symmetries in \( y \) and \( z \) and periodic in \( x \) with up to \( n_x \times n_y \times n_z = 1024 \times 256 \times 2048 \) mesh points. Using these symmetries only one-half of one of the anti-parallel vortices needs to be simulated.

The “symmetry” plane will be defined as \( xz \) free-slip symmetry through the maximum perturbation of the initial vortices and the “dividing” plane will be defined as the \( xy \) free-slip symmetry between the vortices.

II. How should spurious high-wavenumber energy in spectral methods be suppressed?

A generic difficulty in applying spectral methods to localized physical space phenomena is the accumulation of spurious high-wavenumber energy that leads to numerical errors.
Three-dimensional visualization of the singular collapse of anti-parallel vortex tubes in the incompressible Euler equations at $t = 17$. One half of one of the anti-parallel vortices, cut through the symmetry plane of maximum vorticity with $z$ expanded by 4 is shown. This is a a black and white version of the 1996 color cover figure of Nonlinearity [51]. Three visualization procedures are used: mesh lines with shading, an isosurface, and vortex lines. This illustrates how the physical space structure can be divided into three regions, inner, intermediate, and outer by the length scales $R \sim (T_c - t)^{1/2}$ and $\rho \sim (T_c - t)$. The inner region within a distance $\rho \sim (T_c - t)$ of $\|\omega\|_\infty$ is visualized with bright lines. $\|\omega\|_\infty$ is among the brightest lines. The dominant feature is an isosurface set at $0.6\|\omega\|_\infty$ indicating the region out to $R$, the extent of the intermediate region. Beyond the isosurface is an outer region indicated by swirling vortex lines that originate from within the surface.
- Does fluid coming out of symmetry plane imply that waves propagate outward?

- No: waves and fluid move in opposite directions on vortex filaments.
How is collapse preserved?
New work from Barenghi group in Newcastle. Bundles of vortices.
• Wiggles in the stretching rate.
• These snapshots are well-within the time of complete convergence up to \( t=8.75 \).
• Still we were concerned about effects of resolution.
• Effect of spirals changes that opinion.
Maybe my new spiral collapse model will explain this.
Three images of 3D forced $64^3$ turbulence in a box. Lines show vector directions for quantities above a threshold. In all three frames the primary field, vorticity, is shown using blue lines. Left frames are at $t = 1.50$, but different perspectives and different secondary fields. **Yellow** is for scalar gradients and **red** is for compressive strain, **White** is the overlap between the scalar or compressive fields and two interacting regions of vorticity.

**Before** I emphasized the difference in the center between the bottom two frames due to reconnection of the two central vortices.

**Today** I want to emphasize the **spiraling** or **interwoven** vortices just above and below this event.
New vortex model

• Interacting anti-parallel vortex filaments.

  It has two parts:

• Non-local interaction between two anti-parallel, mirrored vortices.

  Using only Biot-Savart interaction near the symmetry plane.

New quaternionic formulation (due to Gibbon) to generate time-derivative of curvature (in Gibbon, Holm, Kerr, Roulstone).

  But this is not enough.

• Need out-of-plane self-contribution. Therefore use:

• Local induction approximation of a filament.

  – This is converted into NLS (Hasimoto transformation) of a wavefunction $\psi$.

  – Whose real and complex parts, with an additional scalar gauge, give the direction and strength of curvature of the filament.
Symmetry plane Biot-Savart

Velocity on the symmetry plane due to a cylindrical vortex of uniform radius (in Rosenhead regularization) is:

\[ u = \frac{\Gamma}{2\pi} \left[ 1 + \frac{\kappa x_n}{2} \right] \left( -y, x \right) + \frac{\Gamma \kappa}{8\pi} \left[ \log \frac{\epsilon^2}{r^2 + a^2} - 2 \right] (n_y, n_x) \]  

which minus the regularization is equation (2.3.9) of Saffman’s book. This neglects effect of twisting (torsion) of vortex lines out of the symmetry plane.

- To close: need equation for \( \kappa = \kappa n \). Use quaternions for Euler.

- From Gibbon (2002), rate-of-change of the direction of vorticity is:

\[ \frac{D}{Dt} \hat{\omega} = \chi \times \hat{\omega} \quad \text{where} \quad \chi = \hat{\omega} \times S \hat{\omega} \]  

- Then take the arclength derivative to get (Gibbon et al., 2006)

\[ \frac{D\kappa}{Dt} = (\chi \times \hat{\omega})_s - \alpha \kappa \]  

- This can be reformulated in several ways. Most useful for filaments:

\[ \frac{D\kappa}{Dt} = u_{ss} - \alpha_s \hat{\omega} - 2\alpha \kappa \]  

- If the complex variable \( \psi \) satisfies \( Re(\psi(s)e^{-i\theta_1}), Im(\psi(s)e^{-i\theta_1}) = \kappa = (\kappa_x, \kappa_y) \)

Then if \( \alpha = 0 \) (as in the local induction approximation) (5) \( \kappa \) is equivalent to the nonlinear Schrödinger equation for \( \psi \) due to the Hasimoto transformation.

\[ \dot{\psi} = i\psi'' + \frac{1}{2}\psi|\psi^2| \]
Hasimoto transformation for curvature in an NLS equation

The Hasimoto transformation is derived from the local induction approximation for motion $\dot{X}$ of a thin vortex filament described by:

$$\frac{\partial X}{\partial t} = G\kappa b$$ (7)

where

$$G = (\Gamma/4\pi)[\log(\epsilon/r) + O(1)]$$ (8)

To which are added the Frenet-Serret equations of differential geometry

$$X' = \hat{\omega}, \quad \hat{\omega}' = \kappa n$$
$$n' = \tau b - \kappa \hat{\omega}, \quad b' = -\tau n$$ (9)

Here $\hat{\omega}$=the direction of vorticity, $\kappa$=curvature, $n$=direction of curvature, $b = \hat{\omega} \times n$ is the bi-normal and $\tau$=torsion where $\kappa n = (\hat{\omega} \cdot \partial_s)\hat{\omega}$.

Then the wave function is

$$\psi(s) = \kappa(s) \exp(i \int_0^s \tau(s')ds' + i\theta_0 + i\theta_1)$$ (10)

- On the symmetry plane: $\kappa \neq 0 \quad \kappa' = 0 \quad \tau = 0 \quad \tau' \neq 0$
- Therefore
  $$\dot{\kappa} = -\kappa\tau' \quad \dot{n} = \frac{\kappa''}{\kappa} b \quad \frac{d}{dt} \kappa n = -\tau' \kappa n + \kappa'' b$$ (11)

- I can show that given $\psi$ and an extra gauge variable $\theta_1(t)$ at one point on the curve that a separate equation for $n$ does not need to be solved. $\dot{\theta}_1 = \frac{1}{2}\kappa^2$. 
Understanding direction of wave/soliton propagation

- For this we need the equation of the torsion derivative \( \frac{d\tau}{ds} \) on the symmetry plane where the NLS wave function is:
  \[ \psi = \sqrt{\kappa} e^{i\int \tau ds} \quad \text{and} \quad v_{ph} = \frac{d}{ds} \int \tau ds = \tau \]

- Sign of torsion determines the direction of phase velocity.

- On the symmetry plane one finds that:
  \[ \frac{d\tau'}{dt} = \frac{\kappa''''}{\kappa} - \left(\frac{\kappa''}{\kappa^2}\right)^2 + \kappa \kappa'' - 2(\tau')^2 \]

  - \( \max(\kappa) \) is on the symmetry plane, therefore \( \kappa'' < 0 \). Negative.
  - \( \max(-\kappa'') < 0 \). Positive definite. Therefore:

  \[ \frac{\kappa''''}{\kappa} - \left(\frac{\kappa''}{\kappa^2}\right)^2 + \kappa \kappa'' - 2(\tau')^2 \]

  Which sign dominates?
Which sign dominates?

- If you have a sharp \textbf{artificial} reconnection, then
  \[ \kappa''' > 0 \] dominates, \( \tau' > 0 \) and waves propagate outwards.

- To get a singularity one needs \( \tau' < 0 \). Why?
  c) It is necessary to add this to the non-local Biot-Savart terms in order to give the curvature a \textbf{tilt}.
  d) If there is no \textbf{tilt} then stretching will saturate
  e) And there won’t be any singularity.

- But if there is stretching along the vortex filament:
  This will be due to the non-local anti-parallel Biot-Savart interaction.
  a) Then the \( < 0 \) term could dominate at late times.
  b) Then waves propagate inwards, colliding on the symmetry plane.
Cylindrically symmetric anti-parallel vortices.

Variables

- \( d \) separation of vortices
- \( a \) radius of vortices
- \( \kappa \) curvature on symmetry plane
- \( \theta_0 \) gauge variable

\[ \int \kappa^2 ds \text{ is conserved by NLS partA} \]

\( \tau' \) is critical for curvature blowup.
What happens after reconnection?

Near future plans

- Of course bigger and better simulations.

- Analysis
  - Scaling analysis
    Does circulation collapse with $\|\omega\|_\infty$?
    Velocity and circulation.
  - Pressure Hessian
  - Acoustic noise generation: Could small scales generated by the nearly singular events be the strongest source of acoustic noise?
References


