Numerical study of 3D Rayleigh–Taylor turbulence

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Rayleigh–Taylor turbulence

- Turbulence initiated by the Rayleigh–Taylor instability

2D Numerical simulation (color-coded density)
3D Numerical simulation (iso-surface of the density)

- Structure functions in the mixing zone (*time-dependent* !)

\[
\left\langle \left( \left[ \mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t) \right] \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \right)^p \right\rangle \propto r^{\xi_p} t^{\mu_p},
\]

\[
\left\langle (\rho(\mathbf{x} + \mathbf{r}, t) - \rho(\mathbf{x}, t))^p \right\rangle \propto r^{\xi_p} t^{\lambda_p}.
\]

- Dimensional analysis (Chertkov 2003)

\[
\left\langle (\delta_r u(t))^p \right\rangle \propto r^{p/3} t^{p/3}, \quad \left\langle (\delta_r \rho(t))^p \right\rangle \propto r^{p/3} t^{-2p/3}.
\]
Mixing zone

- Mixing zone length \( L(t) = C_F \, A \, g \, t^2 \) \hspace{1em} (E. Fermi 1951)

\[
A = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} : \text{Atwood number,} \quad g : \text{gravitational acceleration,} \\
C_F : \text{(universal?) constant}
\]
• Turbulence within the mixing zone
  – Largest length scale evolves as $L(t) = C_F A t^2$
  $\Leftarrow$ Similarity variable $\frac{r}{L(t)}$
  – Boundary effects (top, bottom or side walls) are not yet important.
  $\Leftarrow$ Universality of statistical laws
Scope of this talk

- Numerical study of the 3-D Rayleigh–Taylor turbulence
  - Mixing zone length law \( L(t) = C_F A g t^2 \), does this hold?
  - If \( L(t) \propto t^2 \), the velocity and the density structure functions?

\[
\langle [u(x+r, t) - u(x, t)]^p \rangle \sim r^{\xi_p} t^{\mu_p}, \quad \langle [\rho(x+r, t) - \rho(x, t)]^p \rangle \sim r^{\xi_p} t^{\lambda_p}
\]

Compare them with the phenomenology of Chertkov (2003)

* In 2-D, such comparison was already made (Celani, Mazzino, & Vozella 2006).

* Many other 3D R-T simulations by D. Youngs, Y. Zhou, . . .
Setting

- Incompressible Navier-Stokes eqs. with the variable density $\rho$ and the gravity (acceleration $g$).

\[
\rho \left[ \partial_t u + (u \cdot \nabla) u \right] = -\nabla p + \rho g + \mu \nabla^2 u, \\
\nabla \cdot u = 0, \\
\partial_t \rho + (u \cdot \nabla) \rho = \kappa \nabla^2 \rho.
\]

\[ g = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \]
- Rescaled density $\tilde{\rho}$ ($-1 \leq \tilde{\rho} \leq 1$):

$$\rho = \frac{\rho_1 + \rho_2}{2} + \frac{\rho_2 - \rho_1}{2} \tilde{\rho} = \frac{\rho_1 + \rho_2}{2} (1 + A) \tilde{\rho}$$

- Atwood number

$$A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \quad (0 < A < 1)$$
• Scaled (non-dimensionalized) equations:

\[
(1 + A\tilde{\rho})[\partial_t \tilde{u} + (\tilde{u} \cdot \tilde{\nabla})\tilde{u}] = -\tilde{\nabla}(\tilde{\rho} - A\tilde{g}\tilde{z}) + A\tilde{g}\tilde{\rho} + \frac{1}{g^{1/2}\ell^{3/2}}\tilde{\nabla}^2\tilde{u},
\]

\[
\tilde{\nabla} \cdot \tilde{u} = 0,
\]

\[
\partial_t \tilde{\rho} + (\tilde{u} \cdot \tilde{\nabla})\tilde{\rho} = \frac{1}{g^{1/2}\ell^{3/2}}\tilde{\nabla}^2\tilde{\rho}.
\]

\[
\left( A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \quad \tilde{g} = (0, 0, -1), \quad \nu = \frac{2\mu}{\rho_2 + \rho_1} \right)
\]

• Boussinesq approximation:

at small Atwood number \( A \ll 1 \),

\[
\partial_t \tilde{u} + (\tilde{u} \cdot \tilde{\nabla})\tilde{u} = -\tilde{\nabla}(\tilde{\rho} - A\tilde{g}\tilde{z}) + A\tilde{g}\tilde{\rho} + \frac{1}{g^{1/2}\ell^{3/2}}\tilde{\nabla}^2\tilde{u}.
\]
Simulation details

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= -\nabla p^* + A \ddot{g} \rho + \nu \nabla^2 u, \\
\nabla \cdot u &= 0, \\
\partial_t \rho + (u \cdot \nabla) \rho &= \kappa \nabla^2 \rho.
\end{aligned}
\]

- Atwood number \( A = 0.15, 0.05 \)
- \( \nu = \kappa \) (\( Pr = 1 \))
- Periodic boundary condition
  (Fourier-spectral method)
- Initial condition
  \[
  \begin{aligned}
  \rho : (\text{step functions in } z) + \text{(2D perturbation)} \\
  u &= 0
  \end{aligned}
  \]
- Domain \((2\pi)^2 \times 16\pi, \quad (4\pi)^2 \times 32\pi, \quad 128^2 \times 1024, \quad 256^2 \times 2048\)
- 4-th order Runge Kutta method
3D Numerical simulation (iso-surface of the density)
Simulation results: mixing zone length

- Mixing zone length $L(t) = C_F A g t^2$ ? ($A$: Atwood number)

- Empirical definition of the mixing zone:
  - Mixing zone top $z = z_2(t)$: $\rho(z = z_2) = \rho_2 - 0.01 \times \Delta\rho$
  - Mixing zone bottom $z = z_1(t)$: $\rho(z = z_1) = \rho_1 + 0.01 \times \Delta\rho$
  - Mixing zone length $L(t) = z_2(t) - z_1(t)$

- Simulation is stopped when $L(t) = 0.65L_z$ ($L_z$: system size in $z$).
• Mixing zone length $L(t) = C_F A g t^2$ ? $(A$: Atwood number$)$

\begin{align*}
\langle L(t) \rangle &= c_1 t^2 + c_2 t + c_3
\end{align*}

(Subdominant correction is suggested by theory)

(Ristorcelli and Clark 2004).
- A way to determine the numerical constant $C_F$ (Cabot & Cook 2006)

\[ L(t) = C_F Ag t^2 \Rightarrow C_F = \frac{(\dot{L}(t))^2}{4AgL(t)}. \]

\[ (\dot{L}(t) = [L(t + \Delta t) - L(t)]/\Delta t) \]

- Cabot & Cook (2006) simulation: $C_F \simeq 0.048 (A = 0.5, 3000^3$ grids).
• Constant $C_F$ for two different Atwood number cases

$$L(t) = C_F Ag t^2 \Rightarrow C_F = \frac{(\dot{L}(t))^2}{4AgL(t)}.$$
• Constant $C_F$ for small and large system sizes

$$L(t) = C_F Ag t^2 \Rightarrow C_F = \frac{(\dot{L}(t))^2}{4AgL(t)}.$$
• When \( \frac{(\dot{L}(t))^2}{4AgL(t)} \approx \text{const.} \), we assume that \( L(t) = C_FAgt^2 \) holds.

• “Time inertial range” can be \( 20 \leq t \leq 40 \) (\( A = 0.15 \)).

• Structure functions for the velocity and the density:

\[
\left\langle \left( \frac{\|u(x + r, t) - u(x, t)\|}{r} \right)^p \right\rangle \propto r^{\xi_p} t^{\mu_p}
\]

\[
\left\langle (\rho(x + r, t) - \rho(x, t))^p \right\rangle \propto r^{\xi_p} t^{\lambda_p}
\]
Simulation results: structure functions in mixing zone

\[
\langle \delta_r u(t)^p \rangle = \left\langle \{u(x + r, t) - u(x, t)\} \cdot \frac{r}{r} \right\rangle^p \propto r^{\xi_p} t^{\mu_p}
\]

\[
\langle \delta_r \rho(t)^p \rangle = \left\langle [\rho(x + r, t) - \rho(x, t)]^p \right\rangle \propto r^{\xi_p} t^{\lambda_p}
\]

- Average \( \langle \cdot \rangle \):
  - spatial average within the mixing zone for fixed time
  - average over different initial perturbations (ensemble average)
Simulation results: structure functions in mixing zone

\[ \langle \delta_r u(t)^p \rangle = \langle \{ u(x + r, t) - u(x, t) \} \cdot \frac{r}{r^p} \rangle \propto r^{\xi_p} t^{\mu_p} \]

\[ \langle \delta_r \rho(t)^p \rangle = \langle [\rho(x + r, t) - \rho(x, t)]^p \rangle \propto r^{\xi_p} t^{\lambda_p} \]
R-T turbulence phenomenology (Chertkov 2003)

- Mixing zone length $L(t) \sim Ag t^2$
- Large-scale velocity $u_L \sim \frac{L(t)}{t} \sim \frac{Ag t^2}{t} \sim Ag t$
- Energy input rate at scale $L(t)$
  \[ \epsilon_L(t) \sim \frac{u^3_L}{L(t)} \sim \frac{L^2(t)}{t^3} \sim (Ag)^2 t. \]
- **Adiabaticity assumption**: for spatial scale $r$ is in the inertial range,
  \[ \epsilon_r(t) \sim \epsilon_L(t) \quad (r \ll L). \]
- Dimensional analysis of the velocity increment $\delta_r u(t)$ with $\epsilon_r(t)$, $r$:
  \[
  \delta_r u(t) = \left[ u(x + r, t) - u(x, t) \right] \cdot \frac{r}{r} \sim \epsilon_r(t)^{1/3} r^{1/3} \sim \epsilon_L(t)^{1/3} r^{1/3} \\
  \sim u_L \left( \frac{r}{L(t)} \right)^{1/3} \sim (Ag)^{2/3} r^{1/3} t^{1/3}. \]
R-T turbulence phenomenology (Chertkov 2003)

- Density dissipation rate at spatial scale $r$ in the inertial range:

$$\epsilon_r^{(\rho)}(t) \sim \epsilon_L^{(\rho)}(t) \sim \frac{(\Delta \rho)^2 u_L}{L(t)} \sim (\Delta \rho)^2 t^{-1} \quad (\Delta \rho = \rho_2 - \rho_1)$$

- Dimensional analysis of the density increment $\delta_r \rho(t)$ with $\epsilon_r^{(\rho)}(t)$, $\epsilon_r(t)$, $r$:

$$\delta_r \rho(t) = \rho(\mathbf{x} + r, t) - \rho(\mathbf{x}, t)$$
$$\sim [\epsilon_r^{(\rho)}(t)]^{1/2} \epsilon_r(t)^{-1/6} r^{1/3}$$
$$\sim [\epsilon_L^{(\rho)}(t)]^{1/2} \epsilon_L(t)^{-1/6} r^{1/3}$$
$$\sim \Delta \rho \left( \frac{r}{L(t)} \right)^{1/3}$$
$$\sim \Delta \rho (Ag)^{-1/3} r^{1/3} t^{-2/3}$$

(generalization of the Obukhov–Corrsin scaling).
R-T turbulence phenomenology (Chertkov 2003)

- Dimensional analysis with the adiabaticity assumption:

\[
\begin{align*}
\delta_r u(t) & = \left[ u(x + r, t) - u(x, t) \right] \cdot \frac{r}{r} \\
& \sim u_L \left( \frac{r}{L(t)} \right)^{1/3} \sim (Ag)^{2/3} r^{1/3} t^{1/3}, \\
\delta_r \rho(t) & = \rho(x + r, t) - \rho(x, t) \\
& \sim \Delta \rho \left( \frac{r}{L(t)} \right)^{1/3} \sim \Delta \rho (Ag)^{-1/3} r^{1/3} t^{-2/3}.
\end{align*}
\]

- \( p \)-th order structure function

\[
\begin{align*}
\langle \delta_r u(t)^p \rangle & \sim r^{\xi_p} t^{\mu_p} \\
\langle \delta_r \rho(t)^p \rangle & \sim r^{\xi_p} t^{\lambda_p}
\end{align*}
\]
• Velocity structure function within the mixing zone

\[ \langle \delta_r u_1(t)^p \rangle = \langle [u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{p/3} \]

**fixed time** \( t = 40 \), domain: \((2\pi)^2 \times 16\pi\), grid: \(128^2 \times 1024\).
\[ \langle \delta_r u_1(t)^p \rangle = \langle [u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{p/3} \]

**fixed time** \( t = 40 \), small domain: \((2\pi)^2 \times 16\pi\), grid: \(128^2 \times 1024\).

large domain: \((4\pi)^2 \times 32\pi\), grid: \(256^2 \times 2048\).
• Density structure function for fixed \( t \) within the mixing zone

\[
\langle \delta_r \rho(t)^p \rangle = \langle [\rho(x_1 + r, x_2, x_3, t) - \rho(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{-2p/3}
\]

fixed time \( t = 40 \), domain: \((2\pi)^2 \times 16\pi\), grid: \(128^2 \times 1024\).
\[ \langle \delta_r \rho(t)^p \rangle = \langle [\rho(x_1 + r, x_2, x_3, t) - \rho(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{-2p/3} \]

fixed time \( t = 40 \), small domain: \( (2\pi)^2 \times 16\pi \), grid: \( 128^2 \times 1024 \).

large domain: \( (4\pi)^2 \times 32\pi \), grid: \( 256^2 \times 2048 \).
• Observation for spatial scaling (fixed time $t$)

- For the velocity, $\langle \delta_r u(t)^p \rangle \sim r^{\xi_p} t^{\mu_p}$
  * hard to say something...
  * deviation from the phenomenology $\langle (\delta_r u)^p \rangle \sim r^{p/3}$ may be small.

- For the density, $\langle \delta_r \rho(t)^p \rangle \sim r^{\xi_p} t^{\lambda_p}$
  * power-law behavior is reasonably seen.
  * deviation from the phenomenology $\langle (\delta_r \rho)^p \rangle \sim r^{p/3}$ is indicated.
Density increments over the vertical direction $\delta_z \tilde{\rho}$ (fixed time $t = 40$)

fixed time $t = 40$, domain: $(2\pi)^2 \times 16\pi$, grid: $128^2 \times 1024$.

- Mean density gradient is subtracted: $\tilde{\rho} = \rho(\mathbf{x}, t) - \frac{\Delta \rho}{L(t)} z$.
- Structure function of $\tilde{\rho}$: $\langle (\delta_z \tilde{\rho})^p \rangle$
• Isotropy: density structure functions for $x$, $y$, $z$ directions

- $\delta_x \rho = \rho(X + x, Y, Z, t) - \rho(X,Y,Z,t)$
- $\delta_y \rho = \rho(X, Y + y, Z, t) - \rho(X,Y,Z,t)$
- $\delta_z \tilde{\rho} = \rho(X, Y, Z + z, t) - \rho(X,Y,Z,t) - \frac{\Delta \rho}{L(t)} z$

- In the inertial range, the density fluctuation is roughly isotropic.
- Temporal scaling for **fixed scale** $r$ ($r$ within the inertial range).

Velocity structure function

$$\langle \delta_r u_1(t)^p \rangle = \langle [u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{p/3}$$

$r = 1.03$, domain: $(2\pi)^2 \times 16\pi$, grid: $128^2 \times 1024$.

- Temporal scaling deviates from the phenomenology even for $p = 2$. 

- Temporal scaling deviates from the phenomenology even for $p = 2$. 

Comparison between large and small domains

\[
\langle \delta_r u_1(t)^p \rangle = \langle [u_1(x_1 + r, x_2, x_3, t) - u_1(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{p/3}
\]

\( r = 1.03, \) small domain: \((2\pi)^2 \times 16\pi,\) grid: \(128^2 \times 1024.\)

large domain: \((4\pi)^2 \times 32\pi,\) grid: \(256^2 \times 2048.\)
• Temporal scaling for **fixed scale** $r$ ($r$ within the inertial range).

Density structure function

$$\langle \delta_r \rho(t)^p \rangle = \langle [\rho(x_1 + r, x_2, x_3, t) - \rho(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{-2p/3}$$

$r = 1.03$, domain:$(2\pi)^2 \times 16\pi$, grid:$128^2 \times 1024$.

• Again temporal scaling disagrees with the phenomenology.
• Comparison between large and small domains

\[ \langle \delta_r \rho(t)^p \rangle = \langle [\rho(x_1 + r, x_2, x_3, t) - \rho(x_1, x_2, x_3, t)]^p \rangle \propto r^{p/3} t^{-2p/3} \]

\[ r = 1.03, \text{ small domain: } (2\pi)^2 \times 16\pi, \text{ grid: } 128^2 \times 1024. \]

\[ \text{ large domain: } (4\pi)^2 \times 32\pi, \text{ grid: } 256^2 \times 2048. \]
• Observation for fixed spacial scale $r = 1.03$

- For both the velocity and density,
  - power law behavior is reasonably seen (we focus on $20 \leq t \leq 40$).
  - disagreement with the phenomenology even for $p = 2$ (!)

$$\langle \delta_r u^p \rangle \sim r^{p/3} t^{p/3}, \quad \langle \delta_r \rho^p \rangle \sim r^{p/3} t^{-2p/3}.$$  

Is this discrepancy real?
Anomalous scaling: scalar exponent saturation

- Saturation of the scaling exponent of the density structure function $\langle \delta_r \rho^p \rangle \sim r^{\xi_p \cdot t^{\lambda_p}}$

$\xi_p \to \text{(const.)}, \lambda_p \to \text{(const.)}$ as $p \to \text{large}$

In “statistically steady-state” turbulence setting, such saturation is a well-known property.
Summary and outlook

• Summary
  – Numerical study of 3-D Rayleigh–Taylor turbulence via Boussinesq approximation is performed (small Atwood number).
  – Mixing zone length \( L(t) = C_F A g t^2 \), \( C_F \approx 0.04 \) is obtained, \( C_F \) value consistent to other studies.
  – Structure functions of the velocity \( \langle \delta_r u^p \rangle \sim r^{\xi_p} t^{\mu_p} \) and the density \( \langle \delta_r \rho^p \rangle \sim r^{\xi_p} t^{\lambda_p} \) is calculated.
    Exponents of \( t \) disagrees with the phenomenology (BUG or real?).

• Outlook
  – If the anomalous scaling is real, what is behind?
    Breakdown of the adiabaticity assumption \( \epsilon_L(t) = \epsilon_r(t) (L > r) \)?
  – Kraichnan-model type analysis with the time-dependent mean scalar gradient \( \frac{\Delta \rho}{L(t)} z \sim \Delta \rho t^{-2} z \).
Density $\rho$ (horizontally averaged)

$\Delta \rho$

$L(t)$

$z$

Density $\rho$

(horizontally averaged)