Outline:

• History of Euler calculations.
• Bustamante and Kerr (2008) (our Aussois contribution)
  Numerical tests in new Physica D paper.
  New scaling laws for Euler collapse.
• Lu/Doering solutions and new Schumacher enstrophy analysis.
  Growth of enstrophy in anti-parallel Euler.
  Further spectral tests and new blow-up quantity.
• Another Navier-Stokes flow with an intense event.
  What I tried before to identify an intense event.
  Evidence for same enstrophy blow-up.
Mathematical Issues

- Outstanding modern mathematical problem, $1 million prize:
- For 3D incompressible **Navier-Stokes** with finite energy, etc.
  - Find an example of a singularity of 3D incompressible Navier-Stokes
  - Prove that Navier-Stokes is regular.
- Folk-belief: Navier-Stokes is regular

- Related: 3D incompressible **Euler**, what is known:
  - There is only one unquestionable analytic bound: Beale, Kato, Majda (1984):
    \[
    \int \| \omega \|_\infty dt \to \infty
    \]
    (This has since also been proven using the more robust BMO norm)
  - Folk-belief: Unknown.
  - Constantin, Fefferman, Majda (1996) and Deng, Hou, Yu (2005) have further relations on time integrals of **curvature** and **velocity**:
    \[
    \int \| \nabla \xi \|_\infty^2 dt \quad \int \sup |u|^2 dt \quad \int \| \nabla \xi \|_\infty |u| dt
    \]
Is there a singularity of Euler?

History of the Numerical Search

– Early analytic: Bardos et al. (1976)

• Morf, Orszag & Frisch (1980): Padé approximations to time series expansion of Taylor-Green. Singularity: Yes
• Chorin (1982): Vortex filaments: Singularity: Yes
• Brachet et al. (1983): Direct numerical simulation (DNS) of Taylor-Green. Vortex sheets and complex-time singularities: Singularity: No
• Siggia (1984): Vortex-filament method; became anti-parallel. Singularity: Yes


• Ashurst & Meiron/Kerr & Pumir (1987): Singularity: Yes/No
• Pelz & Gulak (1997): Kida’s high symmetry. \textbf{Singularity: Yes.}
• Grauer, Marliani & Germaschewski (1998): \textbf{Singularity: Yes.}
  – Cichowlas & Brachet: \textbf{Singularity: No.}
  – Pelz & Ohkitani: \textbf{Singularity: No.}
  – Gulak & Pelz: \textbf{Singularity: Yes.}
• Hou & Li (2006): Anti-parallel: \textbf{Singularity: No.}
• Orlandi & Carnevale (2007): Lamb vortices: \textbf{Singularity: Yes.}
• Bustamante & Kerr (2008): Anti-parallel: \textbf{Singularity: Yes.}
Why is this popular again?

• **Kerr, Phys. Fluids (1993)** claimed to have a numerical solution that satisfied all these constraints.

• **Hou, T.Y., & Li, R.** get a different result than Kerr (1993) for:
  – Nearly the same initial condition.  
  – Many more mesh points.
  
  – and a new J. Comp Phys. paper.
  – WHY?

• **Taylor-Green.**

  Marc Brachet should be gearing up to run probably 4 times the resolution in each direction of the last Brachet calculation.


• **Euler 250, Aussois, France, June 2007**
Physica D: Nonlinear Phenomena

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Euler Equations: 250 Years On - Proceedings of an international conference
Aussois, France
18-23 June 2007
Edited by Gregory Eyink, Uriel Frisch, René Moreau and Andrei Sobolevski

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A slice through one vortex at late times.

Vorticity and its stretching.

Circulation can tell us about both resolution limits and the underlying physics.
3D Euler about a 2D Symmetry Plane (our Aussois contribution)

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Initial results from new calculations of interacting anti-parallel Euler vortices are presented with the objective of understanding the origins of singular scaling presented by Kerr (1993) and the lack thereof by Hou and Li (2006). Core profiles designed to reproduce the two results are presented and compared with classical resolution studies and spectral convergence tests. Most of the analysis is on a $512 \times 128 \times 2048$ mesh, with new analysis on a just completed $1024 \times 256 \times 2048$ used to confirm trends. One might hypothesize that there is a finite-time singularity with enstrophy growth like $\Omega \sim (T_c - t)^{-\gamma}$ and vorticity growth like $\|\omega\|_\infty \sim (T_c - t)^{-\gamma}$. The new analysis would then support $\gamma_\Omega \approx 1/2$ and $\gamma > 1$. These represent modifications of the conclusions of Kerr (1993). Issues that might arise at higher resolution are discussed.

PACS numbers: 47.10.A-, 47.11.Kb, 47.15.ki

I Introduction

One definition of solving Euler’s three-dimension incompressible equations [1] is determining whether or not they dynamically generate a finite-time singularity if the initial conditions are smooth, in a bounded domain and have finite energy. The primary analytic constraint that must be satisfied [2] is:

$$\int_0^T \|\omega\|_\infty dt \to \infty$$ (1)

where $\|\omega\|_\infty$ is the maximum of vorticity over all space. To date, Kerr (1993) [3] remains the only fully three-dimensional simulation of Euler’s equations with evidence for a singularity consistent with this and related constraints [4]. Growth of the enstrophy production and stretching along the vorticity, plus collapse of positions, supported this claim [3]. Additional weaker evidence related to blow-up in velocity and collapsing scaling functions was presented later [5].

There is only weak numerical evidence supporting these claims [6,7]. In a recent paper, as described in one of the invited talks of this symposium, Hou and Li (2006) [8] found evidence that the above scenario failed at late times.

This contribution will first comment on four issues raised at the symposium, then present preliminary new results. The four issues are:

- What criteria should be used to determine when numerical errors are substantial?

- What effect do the initial conditions have on singular trends? A cleaner initial condition is proposed.

- We introduce a new approach for determining whether there is singular behavior of the primary properties and the associated scaling. This is applied to both new and old data.

All calculations will be in the following domain: $L_x \times L_y \times L_z = 4\pi \times 4\pi \times 2\pi$ with free-slip symmetries in $y$ and $z$ and periodic in $x$ with up to $n_x \times n_y \times n_z = 1024 \times 256 \times 2048$ mesh points. Using these symmetries only one-half of one of the anti-parallel vortices needs to be simulated.

The “symmetry” plane will be defined as $xz$ free-slip symmetry through the maximum perturbation of the initial vortices and the “dividing” plane will be defined as the $xy$ free-slip symmetry between the vortices.

II. How should spurious high-wavenumber energy in spectral methods be suppressed?

A generic difficulty in applying spectral methods to localized physical space phenomena is the accumulation of spurious high-wavenumber energy that leads to numerical errors.
Crow instability between anti-parallel vortices. Vortices attract

Typically seen in contrails formed by shedding of wing-tip vortices.

Visualization is water vapor condensation in vortex cores.

Reconnection ensues.
Diagram of the interaction of anti-parallel vortices. From an initial condition of anti-parallel vortices separated at their closest approach by $\delta$, if $\nu \neq 0$ there is reconnection that forms new vortices indicated by the dashed curves. However, if $\nu = 0$, a singularity can form when $\delta = 0$ if the vortices are pushed together by the self-induced strain indicated by $e$. 
Steps in idealized antiparallel vortex reconnection taken from a low resolution, low Reynolds number calculation Melander and Hussain, 1989. From top to bottom, the first two frames show the anti-parallel vortex tubes being pushed together by self-interaction through the law of Biot-Savart. The third frame shows that reconnection has progressed to form two new tubes orthogonal to the original tubes. In the bottom frame the new tubes are separating.
Isosurfaces of $\omega$ in 3D and contours on the symmetry plane for the Euler vortex at $t = 0$, $t = 5.625$, $t = 8.125$ and $t = 9.0625$. Isosurfaces and contours are with respect to the maxima, which are printed in the frames. The last three are blown up by a factor of 4 in each direction. Note how the leading edge is folded over in the later stages.
• Note negative circulation region in initial condition.
• This was not in Kerr (1993).
• **BAD**

**GOOD**

• **WHY?**
Numerical controversy

- Do not allow extraneous bits of circulation.
- It gets sucked into interaction region early.
- Numerical dissipation hits it: circulation is not conserved.
- **Worse:** Circulation pops up in reconnecting plane.
- Suppresses singularity.
- Fancy filters don’t help.

Can’t put lipstick on a pig.
 Numerical controversy

Hou & Li (2006)  
Bustamante & Kerr (2008)

- Note negative circulation region in initial condition.
- This was not in Kerr (1993).
- BAD

- WHY?
- No premature reconnection.
- You NEED that bulge in profile.

GOOD
\( \Omega, \Omega_p \) and \( \| \omega \|_\infty = \max(\omega) \)

- Inverted.
- Taken to powers.
- Normalized.

Designed to show time dependencies indicated.

- The assumed power laws are:
  \( \gamma_\Omega = 1/2 \) for \( \Omega \)
  \( \gamma = 3/2 \) for \( \| \omega \|_\infty \)

- \( \gamma = 3/2 \) is found using a two-parameter fit to \( C \) and \( \gamma \) once \( T_c \) is fixed.

- What are plotted are:
  0.00025/\( \Omega^2 \),
  0.005/\( \Omega_p^{2/3} \),
  1/\( \max(|\omega|)^{2/3} \)
  and 1/\( \max(|\omega|) \).
An alternative approach

**Notation:** $E^3 = \Omega^3$ Enstrophy cubed. $E^{1.78} = \Omega^{1.78}$

$\dot{\Omega} \leq C\Omega^3/\nu^3$ is a rigorous Navier-Stokes result. $\dot{\Omega} = C^3\Omega^3/\nu^3 \implies \Omega = \frac{(\nu/C)^{3/2}}{(T - t)^{1/2}}$

($\dot{\Omega} \sim \Omega^{1.78}$ is close to Karman-Howarth result.)

**3D variational problem** Lu, PhD thesis, University of Michigan 2006 under D. Doering
Growth of enstrophy

Rigorous upper bound for viscous flow
Blue highlights region with $\dot{\Omega} = C\Omega^3$.
One event: Needle in a haystack???

**Pointwise analysis**

Conditioned to the maximal pointwise growth of local enstrophy $\Omega = \omega^2$.
Colliding vortex tubes

Isosurfaces of $\omega^2$

Isosurfaces of $\varepsilon$
Three images of 3D forced $64^3$ turbulence in a box. Lines show vector directions for quantities above a threshold.

**Vorticity**: In all three frames the primary field, vorticity is in **blue**.

**Scalar gradients**: Yellow is for scalar gradients.

**Compressive strain**: Red is for compressive strain.

White is the overlap.

- Left frames are at $t = 1.50$, but different perspectives and different secondary fields. Right: $t = 1.55$.

Note the two interacting regions of vorticity.

And the induced compressive fields: yellow, red, and overlap between.

- **Before** I emphasized the difference in the center between the bottom two frames due to reconnection of the two central vortices.

- **Today** I want to emphasize the growth of enstrophy leading up to this event.
What is going on in the flow?

1983: Is it two parallel vortices merging via 2D dynamics?

1985: Is it a Lundgren spiral?

1987: Is it two nearly anti-parallel vortices merging?

* Strong \textit{dissipation} is created in the process.
* Reconnection.
* But what do enstrophy and peak vorticity do?
Today: Better calculations, better analysis than Kerr (1993)

New analysis: Logarithmic time derivative

- Instantaneous two-parameter fitting.
- Consider the new function: \( g(t) = \left( \frac{d}{dt} \log f(t) \right)^{-1} = \frac{\dot{f}}{f} = \frac{1}{\gamma} (T_c - t) \).

- Works if you know \( f \) and \( \dot{f} \). This is possible for:
  
  Enstrophy: \( \Omega = \int dV \omega^2 \)  
  Enstrophy production: \( \Omega_p = \int dV \omega_i e_{ij} \omega_j \)  
  
  Enstrophy: \( y = 0 \) production: TO BE DONE
  \( \Omega_{SP} = \int dA \omega^2 \)  
  \( \Omega_{SPp} = \int dA \omega^2 \)  
  
  peak vorticity: \( \|\omega\|_{\infty} = \text{sup}(|\omega|) \)  
  local vortex stretching: Needs max resolution
  \( \hat{\omega}_i e_{ij} \hat{\omega}_j \)  
  
  quadrant Helicity: \( H_{\frac{1}{4}} = \int dV u \cdot \omega \)  
  Helicity production: Needs pressure
  \( H_p = \int dA \omega (p + \frac{1}{2} u^2) \)  
  
- \( \Omega + \Omega_p \) analysis applied to (with same conclusions):
  
  - New 1024 × 256 × 2048 calculation.  
  - Old 512 × 128 × 192 Chebyshev calculation.
New scaling laws

Left: $T_c$ Predicted singular times
Right: $\gamma_\Omega$ Predicted exponents

**Top:** New data
Resolutions
512 × 64 × 2048 (dashed),
512 × 128 × 2048 (dotted) and
1024 × 256 × 2048 (solid).

**Bottom:** Old data
Kerr (1993) data at highest resolution (solid). Dashed lines denote gaps in data.
In the graphs for the predicted $T_c$, the dash-dotted diagonal lines denote the $T_c = t$ singularity asymptote.

New data $T_c \rightarrow 12.5 - 13$.
Old data $T_c \rightarrow 19.5$
whereas before I estimated $T_c = 18.7$.
For both: $\gamma_\Omega \rightarrow 0.5$
If \( \Omega = \frac{C}{(T_c - t)^\gamma} \) need to fit 3 parameters.

If you know \( \Omega \) and \( \dot{\Omega} \) independently, then

**Use: Logarithmic time derivative**

\[
\Omega(t) = \left( \frac{d}{dt} \log \Omega(t) \right)^{-1} = \frac{\Omega}{\gamma \Omega},
\]

\[
\dot{\Omega} = \frac{1}{\gamma \Omega} (T_c - t).
\]

Instantaneous two-parameter fitting.

Similar to Dlog Pade for time series.
If $\Omega = \frac{C}{(T_c - t)^\gamma}$ need to fit 3 parameters.

If you know $\Omega$ and $\dot{\Omega}$ independently, then

**Use: Logarithmic time derivative**

$$\Omega(t) = \left( \frac{d}{dt} \log \Omega(t) \right)^{-1} = \frac{\Omega}{\dot{\Omega}} = \frac{1}{\gamma} (T_c - t).$$

**Instantaneous two-parameter fitting.**

Similar to Dlog Pade for time series.

$\Omega$, $\Omega_p$ and $\|\omega\|_\infty = \max(\omega)$ inverted, taken to a power and normalized to show possible time dependences indicated by the fits on the left. The assumed power laws are: $\gamma_\Omega = 1/2$ and $\gamma = 3/2$. $\gamma = 3/2$ is used based on two-parameter fits to $C$ and $\gamma$ once $T_c$ is fixed. What are plotted in the upper right are: $0.00025/\Omega^2$, $0.005/\Omega_p^{2/3}$, $1/\max(|\omega|)^{2/3}$ and $1/\max(|\omega|)$. 
• If $\|\omega\|_\infty \approx C_\omega(T_c - t)^{-1.5}$ and $\Omega \approx C_\Omega(T_c - t)^{-0.5}$ then

• For classical Euler that as $t \to T_c$ this implies that vortex cores are strongly deformed, but compact.

• compact means circulation collapses with the structure.

• Robust conclusions:

\[ \gamma_\Omega \approx 0.25 - 0.5, \text{ which is small but: } \Omega \text{ does NOT } \to \ln(T - t) \text{ as in Kerr (1993).} \]

\[ \gamma > 1 \text{ for } \|\omega\|_\infty \text{ (not } \gamma \equiv 1 \text{ as in Kerr (1993))} \]

• Speculative: $\gamma_\Omega = 1/2$ and $\gamma = 3/2$
This is not how I plotted things in 1993.

Note that $1/\sup |\omega| \ldots$ is curved. So $\gamma \neq 1$. 
Numerical convergence tests

Error in the circulation $\sigma_z$ through the dividing plane normalized by the initial circulation of the symmetry plane $\sigma_y$: $\frac{\sigma_z(t) \sigma(0)}{\sigma} \times 10^3$

Top: Resolution study of $||\omega||_\infty$ (left) & symmetry plane enstrophy $\Omega_{SP}$ (right), for $n_x \times n_y \times n_z$ of: $512 \times 64 \times 2048$ (dashed), $512 \times 128 \times 2048$ (dotted) and $1024 \times 256 \times 2048$ (solid).

Bottom: Anisotropic energy spectrum (direction $k_x$) at time $t = 10$ for $1024 \times 256 \times 2048$. Points correspond to numerical data. The solid curve corresponds to the fit of the spectrum according to $\log E(k_x) = C - n \log(k_x) - 2\delta k_x$, where the fit interval is defined by the vertical dashed lines.
Resolution study for $n_x \times n_y \times n_z$ of:
512 $\times$ 64 $\times$ 2048 (dashed), 512 $\times$ 128 $\times$ 2048 (dotted) and 1024 $\times$ 256 $\times$ 2048 (solid).
Anisotropic energy spectrum (direction $k_x$) at time $t = 10$ for $1024 \times 256 \times 2048$. Points correspond to numerical data. The solid curve corresponds to the fit of the spectrum according to $\log E(k_x) = C - n \log(k_x) - 2\delta k_x$, where the fit interval is defined by the vertical dashed lines.

Error in the circulation $\sigma_z$ through the dividing plane normalized by the initial circulation of the symmetry plane $\sigma_y$:

$$\frac{\sigma_z(t) - \sigma_z(0)}{\sigma_y(0)} \times 10^3$$
In the $x - z$ symmetry plane:

- $\omega$ vorticity (grey scale) and
- $\alpha$ vortex stretching (solid and dashed contours).
In the $x - z$ symmetry plane:

- $\omega$ vorticity (grey scale) and
- $\alpha$ vortex stretching (solid and dashed contours).

- Note that $\max(\alpha)$ is at $x = 28$ but $\max(\omega)$ is at $x = 16$.
- Minimum: $\min(\alpha)$ is at a local
  $\max(|\nabla \omega|)$
  $\sim \max(|\Delta \mathbf{u}|)$.
- $\max(\omega)$ is near
  $\max(|\nabla \alpha|)$.

This makes finding $\alpha$ at position of $\|\omega\|_\infty$ nearly impossible, so that two-parameter logarithmic fitting of $\|\omega\|_\infty$ power law is impossible.
Symmetry plane enstrophy $\Omega_{SP}$ blow-up bounds

- Now to try something really dangerous (for me at least).
- Try some real mathematics.
- Can we create an Euler version of the growth of enstrophy bound for Navier-Stokes?
- That is the $\dot{\Omega} \leq C\frac{\Omega^3}{\nu^3}$ bound that underlies the solutions of Lu and Doering.

- Growth of enstrophy in the symmetry plane ($\frac{d}{dt}\Omega_{SP}$)
  - Numerics: Recall that $\Omega_{SP}$ grows strongly.
  - I have an argument why.
  - It suggests a power law.
  - Circulation normalizations:
    These imply normalizations based upon circulation for both $\Omega_{SP}(t)$ and $\Omega(t)$.
  - The power law fits.
  - The circulation normalizations work.
Symmetry plane equations

In the symmetry plane we have:

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u}_h \cdot \nabla) \omega = - (\nabla \cdot \mathbf{u}_h) \omega
\]  

or

\[
\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \mathbf{u}_h) = 0
\]

Implying that vorticity in the symmetry plane acts as a density.

For enstrophy we have

\[
\frac{1}{2} \frac{\partial \omega^2}{\partial t} + \frac{1}{2} (\mathbf{u}_h \cdot \nabla) \omega^2 = - (\nabla \cdot \mathbf{u}_h) \omega
\]  

or

\[
\frac{\partial \omega^2}{\partial t} + \nabla \cdot (\omega^2 \mathbf{u}_h) = \alpha \omega^2
\]

Because on the symmetry plane \( \nabla \cdot \mathbf{u}_h + \alpha = \nabla \cdot \mathbf{u} = 0 \) \( \Rightarrow \ - \nabla \cdot \mathbf{u}_h + \alpha \)

In vortex cores \( \alpha > 0 \), so enstrophy is growing there.

Conservation laws

a: Conservation of circulation: by (3) \[ \int dA \omega = \Gamma \]  

b: Conservation of mass: \[ \int dA \alpha = 0 \]
• Because (2) is a 2D equation of the same form as 2D Euler, the plan now is start
by repeating as much of 2D analysis as possible

• Young’s inequality in arbitrary dimension with: $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$, $\beta > 0$

$$ab \leq \frac{\beta^p a^p}{p} + \frac{b^q}{\beta^q q} \quad (7)$$

• Galiardo-Nirenberg inequality for two-dimensions:

$$\left| \int_{\Delta A} u \cdot \nabla u \cdot \Delta u d^2x \right| \leq c \|\Delta u\|_2 \|\nabla u\|_2^2 \quad (8)$$

• Apply Young’s inequality to get

$$\left| \int_{\Delta A} u \cdot \nabla u \cdot \Delta u d^2x \right| \leq \frac{c}{2} \left( \beta \|\Delta u\|_2^2 + \frac{1}{\beta} \|\nabla u\|_2^4 \right) \quad (9)$$

• The objective now is to isolate the $\beta^{-1}\|\nabla u\|_2^4 = \beta^{-1}\Omega^2$ term to get an inequality in terms of easily understood, and often conserved, norms.

• But to do this we need to eliminate the $\beta\|\Delta u\|_2^2$ palinstrophy term.

• In Navier-Stokes this is possible because there is a negative definite depletion term, viscosity, of the same form.

So $\beta$ is chosen to cancel this: $\beta = 2\nu/c$

• The question now to to find a symmetry plane depletion term that can serve the same purpose.
• **depletion** seems possible due to conservation of mass (6a)

⇒ RHS of $D_t(\omega^2) = \alpha \omega^2$ (5) has both positive and negative regions.

We want a lower bound on the integral of the depletion: $\int dA(\alpha_-)\omega^2 \leq C < 0$

where $\alpha_- = \alpha$ for $\alpha < 0$ and $\alpha_- = 0$ for $\alpha \geq 0$
• Depletion seems possible due to conservation of mass (6a)

⇒ RHS of $D_t(\omega^2) = \alpha \omega^2$ (5) has both positive and negative regions.

We want a lower bound on the integral of the depletion: 

$$\int dA(\alpha_-) \omega^2 \leq C < 0$$

where $\alpha_- = \alpha$ for $\alpha < 0$ and $\alpha_- = 0$ for $\alpha \geq 0$

• How to make an estimate?

The relevant regions are within the region of collapsed vorticity, but not at the maximum vorticity or stretching.
• **depletion** seems possible due to conservation of mass (6a)

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• How to make an estimate?

The relevant regions are within the region of collapsed vorticity, but not at the maximum vorticity or stretching.

• Therefore, we should be able to use **estimates of the average** vorticity and stretching within the collapsed region based on a characteristic length scale.
Proposal: The relevant inverse length scale should go as \( r^{-2} \sim \frac{\| \nabla u \|_2^2}{\| \nabla u \|_2^2} \)
so that where \( \alpha < 0 \) and \( \omega \sim r^{-2} \Gamma \): \( \alpha \sim -r^{-2} \Gamma \)

Then \( \int \, dA \, (\alpha \omega^2 \sim -\frac{\Gamma}{r^2} \| \omega \|_2^2 \sim -\Gamma \frac{\| \nabla u \|_2^2}{\| \nabla u \|_2^2} \| \omega \|_2^2 \sim -\Gamma \| \nabla u \|_2^2 \) \)

Now combine with (9) to get an estimate of the production term in (5)
\[
\frac{D}{D t} \| \omega_{SP} \|_2^2 \leq -\Gamma \| \nabla u \|_2^2 + \frac{c}{2 \beta} \left( \beta \| \triangle u \|_2^2 + \frac{1}{\beta} \| \nabla u \|_2^4 \right)
\]

Now choose \( \beta = 2\Gamma/c \) in the standard way to eliminate the palinstrophy terms and we are left with
\[
\frac{D}{D t} \| \omega_{SP} \|_2^2 \leq \frac{c^2}{4\Gamma} \| \nabla u \|_2^4
\]
or in layman’s language
\[
\frac{D}{D t} \Omega_{SP} \leq C \frac{\Omega_{SP}^2}{\Gamma}
\]
But we do not continue as in 2D Navier-Stokes because:

- There is no Poincaré inequality to bound the time integral of enstrophy.

For 2D Navier-Stokes

\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{u} \|^2_2 = -\nu \| \nabla \mathbf{u} \|^2_2 \Rightarrow
\]

\[
\int_0^t \| \nabla \mathbf{u} \|^2_2 ds = \frac{1}{2\nu} (\| \mathbf{u}(0) \|^2_2 - \| \mathbf{u} \|^2_2) \leq \frac{1}{2\nu} \| \mathbf{u}(0) \|^2_2 \tag{15}
\]

Apply this to the enstrophy bound in (13) with Gronwall to bound everything else and get regularity of 2D Navier-Stokes.

- This time, physically, energy in the symmetry plane is decreasing, but does not depend in any way on enstrophy, so there are no additional restrictions.

Therefore, in the symmetry plane if the inequality is ever an equality

\[
\frac{D}{Dt} \Omega_{SP} \equiv C \frac{\Omega_{SP}^2}{\Gamma} \text{ then } \Omega_{SP} = \frac{\Gamma/C}{(T-t)} \tag{16}
\]

- Then by analogy, replacing \( \nu \) by \( \Gamma \) for 3D Euler it suggests that:

\[
\Omega \sim \frac{(\Gamma/c)^{3/2}}{(T-t)^{1/2}}
\]
• Enstrophy, both on the symmetry plane $\Omega_{SP}$ and over all of 3D space $\Omega$

• Appropriately scaled and normalized:

  - argument in the symmetry plane $\frac{d}{dt} \Omega_{SP} \leq C \frac{\Gamma}{\Omega_{SP}}$ implies $\frac{0.4\Gamma}{\Omega_{SP}} \sim (T - t)$ (dash)
Enstrophy, both on the symmetry plane $\Omega_{SP}$ and over all of 3D space $\Omega$

- Appropriately scaled and normalized:
  - argument in the symmetry plane $\frac{d}{dt} \Omega_{SP} \leq C \frac{\Gamma}{\Omega_{SP}}$ implies $\frac{0.4\Gamma}{\Omega_{SP}} \sim (T - t)$ (dash)
  - and then an analogy over 3D space $\frac{d}{dt} \Omega \leq C \frac{\Gamma}{\Omega}$ implies $\frac{\Gamma^2}{\Omega^2} \sim (T - t)$ (line)
• Enstrophy, both on the symmetry plane $\Omega_{SP}$ and over all of 3D space $\Omega$

• Appropriately scaled and normalized:

  - argument in the symmetry plane $\frac{d}{dt} \Omega_{SP} \leq C \frac{\Gamma}{\Omega_{SP}}$ implies $\frac{0.4\Gamma}{\Omega_{SP}} \sim (T - t)$ (dash)

  - and then an analogy over 3D space $\frac{d}{dt} \Omega \leq C \frac{\Gamma}{\Omega}$ implies $\frac{\Gamma^2}{\Omega^2} \sim (T - t)$ (line)

\[ \text{Growth of enstrophy and sym-plane } \Omega_0 \]

\[ \text{extrapolate} \ldots \]

\[ \ldots \text{are extrapolations of linear fits at last completely reliable time (} t \approx 10 \text{) to show convergence towards the estimated singular time found earlier.} \]
power law fits

\[ T_c = 13 \quad n_0 = -2.5 \]

\[ n = -n_0 - a(T_c - t)^2 \]
Summary
References


