Random Walks in Random Environments

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December 2008
RWRE in $\mathbb{Z}^d$

An *environment* is a collection of probability distributions indexed by sites: $\omega = \{\omega(x, y)\}_{x, y \in \mathbb{Z}^d}$, such that

$$\sum_{y \in \mathbb{Z}^d} \omega(x, y) = 1, \quad \forall x \in \mathbb{Z}^d.$$ 

Fix an environment $\omega$. $X_n$ a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = x + y | X_n = x) := \omega(x, y).$$

In Simple Random Walk, $\omega(x, y) = \omega(y)$. For $d \geq 2$, not reversible, i.e. not random conductance $P$-law of environment. For most of talk today: $d \geq 2$, $\{\omega(x, \cdot)\}_{x \in \mathbb{Z}^d}$ independent, identically distributed (i.i.d.).
An \textit{environment} is a collection of probability distributions indexed by sites: $\omega = \{\omega(x, y)\}_{x, y \in \mathbb{Z}^d}$, such that

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An environment is a collection of probability distributions indexed by sites: \( \omega = \{ \omega(x, y) \} \), such that

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\( P \)-law of environment. For most of talk today:

\( d \geq 2, \{ \omega(x, \cdot) \} \) independent, identically distributed (i.i.d.).
RWRE is **nearest neighbor** and **uniformly elliptic**: 
\[ \omega(x, y) \in [\kappa, 1 - \kappa] \] for all \(|y| = 1\), and \(= 0\) otherwise \((\kappa > 0)\).

**Goals**: Law of Large Numbers, Central Limit Theorems, location of exits from large balls.  
**Homogenization** - large scale behavior same as that for appropriate simple random walk with **effective** fixed transition \(\bar{\omega}(y)\).

Many surprising phenomena!

Large Deviations: not today; Comets - Gantert - Z. ’00 \(d = 1\). Zerner ’98; Varadhan ’02; Rassoul-Agha ’03; Kosygina - Rezakhanlou - Varadhan ’05; Yilmaz ’08; Peterson ’08
Standing assumption and goals

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Consider ball $B_R = \{ x \in \mathbb{Z}^d : |x|_2 \leq R \}$. Define exit measure

$$\Pi_R(y) = P_\omega(\text{RWRE exits } B_R \text{ at } y).$$

Let $\pi_R$ denote same quantity for simple random walk.

Assume $|\omega(x, e) - \frac{1}{2d}| < \epsilon$ & law of environment is isotropic, $\epsilon$ small.

**Theorem (Bolthausen-Z. ’07, ’09)**

$(\Pi_R - \pi_R)$, *smoothed over distances that grow to infinity with $R$ arbitrarily slowly, converges to 0 in variation norm.*

Holds for $d \geq 3$ (published) and $d = 2$ (harder, in progress).
Summary of recent results

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Homogenization for PDE’s

With operator

\[
L_\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \left( \frac{x}{\epsilon}, \omega \right) \partial_{x_i x_j}^2 + \sum_{i=1}^{d} \frac{1}{\epsilon} b_i \left( \frac{x}{\epsilon}, \omega \right) \partial_{x_i}
\]

\(a, b\) smooth, finite range dependence, isotropic law.

Set, on \((0, \infty) \times \mathbb{R}^d\)

\[
\partial_t u_\epsilon = L_\epsilon u_\epsilon + g, \quad u_\epsilon(0, \cdot) = f.
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Theorem (Sznitman, Z. ’06)

\(u_\epsilon\) converges uniformly over compacts to the solution of

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\partial_t u_0 = \frac{\sigma^2}{2} \Delta u_0 + g, \quad u_0(0, t) = f.
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With no \( 1/\epsilon \) before drift, homogenization proved by Yurinskii (1980), based on Alexandrov-Bakelman-Pucci \( L^q \) estimates, \( 1 < q < 1 + 1/d \), on invariant measure (also Papanicolau-Varadhan,....).
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$d = 1$: properties

Reversible model.


Ballistic regime can still be heavy tailed (stable).

Quenched CLT, when exists, only with random centering.
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$d \geq 2$: Traps

New challenges: not reversible implies no easy transience criterion.

Role of traps changes: first, they can be avoided. Second, they are (probably) weaker: size is volume controlled, effect is diameter controlled.

**Heuristics:**

$d = 1$: trap can cause exit time from small region at distance $\epsilon n$ from origin, to be of order $\Omega(n)$ and even $\gg n$.

$d > 1$: trap can only cause exit time to be of order $o(n)$.

$\Rightarrow$ No obvious strategy for walk to remain trapped so much that subdiffusive behaviour occurs. Maybe does not occur?
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**d ≥ 2: Traps**

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\(\Rightarrow\) No obvious strategy for walk to remain trapped so much that subdiffusive behaviour occurs. Maybe does not occur?
d ≥ 2: 0-1 law

Embarrassing open problem: 0-1 law.

For $\ell \in S^{d-1}$, set $A_\ell^{\pm} := \{X_n \cdot \ell \to \pm \infty\}$. Prove that $\mathbb{P}(A_\ell^+) \in \{0, 1\}$.

Known - $\mathbb{P}$ i.i.d.:

- $\mathbb{P}(A_\ell^+ \cup A_\ell^-) \in \{0, 1\}$ Kalikow '81.
- 0-1 true for $d = 2$ Zerner-Merkl '02.
- On $A_\ell$, deterministic speed Szmitman-Zerner '99, Zerner '02
- 0-1 law $\Rightarrow$ LLN (possibly 0 speed) Zerner '02
- At most two possible limiting values of $X_n/n$ Varadhan '03 by large deviations; Berger '06.
- If two limiting values and $d \geq 5$, one must be zero Berger '06.
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For $\ell \in S^{d-1}$, set $A^{\pm}_\ell := \{ X_n \cdot \ell \to \pm\infty \}$. Prove that $\mathbb{P}(A^+_\ell) \in \{0, 1\}$.

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For $\ell \in S^{d-1}$, set $A_{\ell}^{\pm} := \{X_n \cdot \ell \to \pm \infty\}$. Prove that $\mathbb{P}(A_{\ell}^{+}) \in \{0, 1\}$.

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But: $0 - 1$ law not true for certain ergodic environments: escape in different directions, positive speed.

Zerner-Merkl '02 $d = 2$, not unif. elliptic.
$d \geq 3$, Bramson-Z.-Zerner '05, uniformly elliptic, mixing environment.
$d \geq 2$: breakdown of 0-1 law

But: 0 – 1 law not true for certain ergodic environments: escape in different directions, positive speed.

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\(d \geq 2: \text{ballistic behavior and regenerations}\)

**Assume**: \(\mathbb{P}(A_{\ell}) = 1\) (escape to \(\infty\) in \(\ell\) direction)

Implies existence of special points where walk does not backtrack (regeneration times)

Plot \(X_n \cdot e_1\):

\[\text{Diagram showing the plots at } \tau_1 \text{ and } \tau_2\]
$d \geq 2$: ballistic behavior and regenerations

**Assume:** $\mathbb{P}(A_\ell) = 1$ (escape to $\infty$ in $\ell$ direction)

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Regeneration times imply LLN because of i.i.d. structure
Sznitman-Zerner, Zerner ’99, ’02

Sufficient conditions that imply inter-regeneration times have all moments, and therefore CLT Sznitman ’99–’05

Under these conditions, CLT also holds quenched (i.e. conditioned on environment)
QCLT=ACLT (no random centering)
Rassoul-Agha and Seppäläinen ’07, Berger-Z. ’07
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\(d \geq 2: \) ballistic behavior and regenerations

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$d \geq 6$: non ballistic behavior

Can replace regeneration times (in $d \geq 5$) by **cut times**
$d \geq 6$: non ballistic behavior

Can replace regeneration times (in $d \geq 5$) by \textbf{cut times}
$d \geq 5$: non ballistic behavior

For simple random walk, density of cut times for $d \geq 5$ is positive (Erdős and Taylor, 1960). If RWRE has $d_0 \geq 5$ dimensions in which it is a SRW, can use the cut times to generate independence.

Theorem (Bolthausen-Sznitman-Z. ’03)

*With above assumptions, get LLN (when $d_0 \geq 5$), CLT when $d_0 \geq 7$, and CLT (quenched, non-random centering) when $d_0 \geq 13$.*

Further, there are examples (even in perturbative regime) where $E_P(\text{drift} \cdot e_1) \geq 0$ but $X_n \cdot e_1/n \to -v_1 < 0!$

Static measure $\neq$ dynamic measure (viewed from the particle’s point of view)

$\Rightarrow$ results in summary cannot be true in general without isotropy assumption
$d \geq 5$: non ballistic behavior

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$\Rightarrow$ results in summary cannot be true in general without isotropy assumption
**$d \geq 5$: non ballistic behavior**

For simple random walk, density of cut times for $d \geq 5$ is positive (Erdős and Taylor, 1960). If RWRE has $d_0 \geq 5$ dimensions in which it is a SRW, can use the cut times to generate independence.

**Theorem (Bolthausen-Sznitman-Z. ’03)**

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Homogenization - perturbative regime

Two major (related) complications in applying homogenization technique to general RWRE model:

- Existence of local drift.
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- Small disorder $d \geq 3$: $\omega(0, e) = \frac{1}{2d} + \varepsilon k(0, e)$
  $P$ i.i.d. invariant under $\mathbb{Z}^d$-rotations.

$\implies$ CLT Bricmont-Kupianien ’91

Proof uses (hard) renormalization techniques, hard to penetrate.

Sznitman-Z ’06 present, for diffusions in random environments, an alternative, somewhat clearer proof to BK
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Exit measures - isotropic perturbative regime

Goal: investigate exit measure from ball $B_R = \{ z \in \mathbb{Z}^d : |z|_2 \leq R \}$.

\[ \Pi_R(A) = P_\omega (\text{RWRE exits } B_R \text{ through } A). \]

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Theorem (Bolthausen-Z ’07)

(d \geq 3) For any $\delta > 0$ there is an $\epsilon_0$ such that with perturbations $\epsilon < \epsilon_0$ it holds that

\[ \limsup_{R \to \infty} \| \Pi_R - \pi_R \|_{T.V.} < \delta \]

and

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Further, the RWRE is transient.

Here $\tilde{\pi}_s$ is exit law from a ball of random radius of order $s$. 
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Exit measures - isotropic perturbative regime - proof

**Scales** $L_{n+1} = L_n \log(L_n)^3$.

A point $x$ is **bad** at level $i$, $i = 1, 2, 3$, if

$$\| \Pi_{L_n} - \pi_{L_n} \|_{T.V.} < \delta$$

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$$(\log L_n)^{-9 + \frac{9(i-1)}{4}} \leq \| \Pi_{L_n} \star \tilde{\pi}_{L_n} - \pi_{L_n} \star \tilde{\pi}_{L_n} \|_{T.V.} < (\log L_n)^{-9 + \frac{9i}{4}}.$$ 

It is bad at level 4 if either

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Basic induction:

\[ P(0 \text{ is } i \text{ bad}) \leq \frac{1}{4} \exp \left[ -\left(1 - \frac{4 - i}{13}\right)(\log L_n)^2 \right]. \]

Theorem is that induction hypothesis on \( n \leq n_0 \) propagates to \( n_0 + 1 \).
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Exit measures - propagation of smoothed estimate

Coarse grain walk, keeping coarse graining all the way to the boundary (slightly finer scale near boundary, but still mesoscopic).

- There is only one bad box.
- If no bad box: Perturbation expansion.

\[ \prod_{L_{n+1}} * \tilde{\pi}_{L_{n+1}} - \pi_{L_{n+1}} * \tilde{\pi}_{L_{n+1}} = \sum \cdots \sum g_{L_n}(0, y) \Delta^{k_1}(y, y') \pi_{L_n}(y', y_1) \cdots g_{L_n} \Delta^{k_j} \pi_{L_n} \tilde{\pi}_{L_n} \]

where \( g_L \) is green function for (coarse grained) SRW, and \( \Delta \) is difference between (coarse grained) RWRE and SRW.

Linear term \((j = 1)\) is most delicate.
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$$\sum g(0, y) \Delta^k(y, y') \pi(y', z) \tilde{\pi}_L$$

Separate according to location of $y$, using everywhere the smoothing $\Delta \pi$ on which induction hypothesis gives information.

**Main term:** $k = 1$, $y$ in bulk.

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$[\cdot - \cdot]$ of order $L_k / L_{k+1}$, but $\sum g(0, y) = (L_{k+1} / L_k)^2$ - not good!

Expand to second order. Expected value vanishes because of isotropy and fact that exit probability is harmonic function.

To get estimate of non-averaged term, use that sum is over essentially independent variables, and $d \geq 3$. 
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For coarse grained walk, number of terms in expansion is \( \ell^d_k \).

Each (after mean has been substracted) is zero mean and of order

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Ca_k \ell_k^{-(d-2)} \ell_k^{-(d-1)} = a_k \ell_k^{-(d-1)}
\]

where \( a_k \) is standard deviation at scale \( L_k \).

Standard deviation of sum is \( a_{k+1} \sim Ca_k \cdot \ell_k^{d/2-d+1} \).

When \( d \geq 3 \): \( a_{k+1} \ll a_k \).

Recent \( d = 2 \) by controlling constant \( C < 1 \).
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- \( k = 1, \ y \) near boundary: use hitting estimates.
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- handle the single bad box if present by deriving rough Green function
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Use perturbation expansion, refining scale all the way to the boundary.

- bulk errors smoothed by $\pi$ following it and induction hypothesis on smoothed errors.
- There are many “bad” boundary boxes, however have good hitting estimates of them.
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