Complex reflection groups and their associated braid groups and Hecke algebras

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Let $K$ be a characteristic zero field and let $V$ be an $r$-dimensional $K$–vector space. Let $S$ be the symmetric algebra of $V$. Each choice of a basis $(v_1, v_2, ..., v_r)$ of $V$ determines an identification of $S$ with a graded polynomial algebra $S \cong K[v_1, v_2, ..., v_r]$ with $\deg v_i = 1$.

Let $G$ be a finite subgroup of $\text{GL}(V)$. The group $G$ acts on the algebra $S$, and we let $R := S^G$ denote the subalgebra of $G$–fixed polynomials.
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Reflection groups, braids, Hecke algebras
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S \cong K[v_1, v_2, \ldots, v_r] \quad \text{with} \quad \deg v_i = 1.
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Let $K$ be a characteristic zero field and let $V$ be an $r$–dimensional $K$–vector space. Let $S$ be the symmetric algebra of $V$. Each choice of a basis $(v_1, v_2, \ldots, v_r)$ of $V$ determines an identification of $S$ with a graded polynomial algebra

$$S \simeq K[v_1, v_2, \ldots, v_r] \quad \text{with} \quad \deg v_i = 1.$$ 

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\[ P := K[u_1, u_2, \ldots, u_r] \text{ with } \deg u_i = d_i \]

and an integer \( m \), such that

\[ S = K[v_1, v_2, \ldots, v_r] \]

\[ \text{free of rank } m|G| \]

\[ R = S^G \]

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In general $R$ is NOT a polynomial algebra, but there exists a graded polynomial algebra

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not free unless...

free of rank $m|G|$

$$R = \mathcal{S}^G \leftarrow \text{not a polynomial algebra unless...}$$

free of rank $m$

$$P = K[u_1, u_2, \ldots, u_r]$$
Moreover,

\[ m \mid G = d_1 d_2 \cdots d_r \]

As a \( PG \)-module, we have \( S \cong (PG)_{m} \).

Example. Consider \( G = \{ (1 0 \ 0 1), (-1 0 \ 0 -1) \} \subset GL_2(K) \).

Denote by \((x, y)\) the canonical basis of \( V = K^2 \).

Then \( S = K[x, y] \) not free but \( \mathbb{P} = K[x^2, y^2] \) free of rank 2.
Moreover,

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Consider \( G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset \text{GL}_2(K) \).
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Denote by \((x, y)\) the canonical basis of \( V = K^2 \). Then

\[
S = K[x, y] \quad \text{not free}
\]

\[
P = K[x^2, y^2] \quad \text{free of rank 4}
\]

\[
R = S^G = K[x^2, y^2] \oplus K[x^2, y^2]xy \quad \text{free of rank 2}
\]
A finite reflection group (abbreviated frg) on \( K \) is a finite subgroup of \( \text{GL}_K(V) \) (\( V \) a finite dimensional \( K \)-vector space) generated by reflections, i.e., linear maps represented by:

\[
\begin{bmatrix}
\zeta & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}
\]

A finite reflection group on \( \mathbb{R} \) is called a Coxeter group. A finite reflection group on \( \mathbb{Q} \) is called a Weyl group.
A finite reflection group (abbreviated frg) on $K$ is a finite subgroup of $GL_K(V)$ ($V$ a finite dimensional $K$–vector space) generated by reflections, i.e., linear maps represented by

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A finite reflection group on $\mathbb{R}$ is called a Coxeter group.

A finite reflection group on $\mathbb{Q}$ is called a Weyl group.
Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let $G$ be a finite subgroup of $\text{GL}(V)$ ($V$ an $r$-dimensional vector space over a characteristic zero field $K$). Let $S$ denote the symmetric algebra of $V$, isomorphic to the polynomial ring $K[v_1, v_2, \ldots, v_r]$.

The following assertions are equivalent.

1. $G$ is generated by reflections.
2. The ring $R := S^G$ of $G$–fixed polynomials is a polynomial ring $K[u_1, u_2, \ldots, u_r]$ in $r$ homogeneous algebraically independant elements.
3. $S$ is a free $R$–module.

In other words, unless $m = 1$, i.e., $R = P$.
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In other words, unless... $m = 1$, i.e., $R = P$. 
\[ S = K[v_1, v_2, \ldots, v_r] \]
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Examples

For $G = S_r$, one may choose

$u_1 = v_1 + \cdots + v_r$

$u_2 = v_1 v_2 + v_1 v_3 + \cdots + v_{r-1} v_r$

\[ \cdots \]

$u_r = v_1 v_2 \cdots v_r$

For $G = \langle e^{2\pi i/d} \rangle$, cyclic group of order $d$ acting by multiplication on $V = \mathbb{C}$, we have $S_r = K[x]$ and $R = K[x^d]$.

$S_r$ acts naturally on $V = \mathbb{C}^r = \bigoplus \mathbb{C}v_i$.

Fix $d \geq 2$.

For each coordinate consider the reflection $v_i \mapsto \zeta_d v_i$.

We obtain the wreath product $\mathbb{C}^d \wr S_r$, generated by reflections.

This group is called $G(d, 1, r)$.

For each divisor $e$ of $d$, there is a normal reflection subgroup $G(d, e, r)$ of $G(d, 1, r)$ of index $e$.
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\begin{align*}
    u_1 &= v_1 + \cdots + v_r \\
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- For $G = \mathfrak{S}_r$, one may choose

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For each divisor $e$ of $d$, there is a normal reflection subgroup $G(d, e, r)$ of $G(d, 1, r)$ of index $e$. 
Let $G \leq SL_2(C)$ be finite and $g \in G$. Let $\zeta$ be an eigenvalue of $g$. Then $\zeta - 1$ $g$ is a reflection. So, if $G = \langle g_1, \ldots, g_r \rangle$, the group $\langle \zeta - 1 g_1, \ldots, \zeta - 1 g_r \rangle$ is an $frg$.

Note that for $G$ irreducible, we have $G / Z(G) \in \{D_n, A_4, S_4, A_5\}$. For example, the group $G := \langle (1 \ 0 \ 0 \ \zeta^3), \sqrt{-3} 3(\zeta^3 3) \rangle \leq GL_2(Q(\zeta^3))$, with $\zeta^3 := \exp(2\pi i / 3)$, is a $frg$ of order 72, denoted $G_5$, isomorphic to $SL_2(3) \times C_3$.

We may choose $u_1 := v_6^1 + 20 v_3^1 v_3^2 - 8 v_6^2$, $u_2 := 3 v_3^1 v_9^2 + 3 v_6^1 v_6^2 + v_9^1 v_3^2 + v_12^2$, with degrees $d_1 = 6$, $d_2 = 12$ (note that $d_1 d_2 = 72 = |G|$).

If $g \in SL_3(C)$ is an involution, then $-g$ is a reflection. Note that $A_5$, $PSL_2(7)$ and $3A_6$ have faithful 3-dimensional representations and are generated by involutions.
Let $G \leq \text{SL}_2(\mathbb{C})$ be finite and $g \in G$. 

So, if $G = \langle g_1, \ldots, g_r \rangle$, the group $\langle \zeta - 1 \rangle g_1, \ldots, \zeta - 1 g_r \rangle$ is an $r$-frg. 

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For example, the group $G := \langle (1 0 0 \zeta_3^3), \sqrt{-3} \rangle \langle -\zeta_3^3 \zeta_2^2 3^2 \zeta_2 \rangle \rangle \leq \text{GL}_2(Q(\zeta_3))$, with $\zeta_3 := \exp(2\pi i/3)$, is a $72$-frg, denoted $G_5$, isomorphic to $\text{SL}_2(3) \times \mathbb{C}^3$. 

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For example, the group

$$G := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle \leq \text{GL}_2(\mathbb{Q}(\zeta_3)),$$

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$$G := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle \leq \text{GL}_2(\mathbb{Q}(\zeta_3)),$$

with $\zeta_3 := \exp(2\pi i/3)$, is a frg of order 72, denoted $G_5$, isomorphic to $\text{SL}_2(3) \times \mathfrak{S}_3$.

We may choose

$$u_1 := v_1^6 + 20v_1^3v_2^3 - 8v_2^6, \quad u_2 := 3v_1^3v_2^9 + 3v_1^6v_2^6 + v_1^9v_2^3 + v_2^{12},$$
Let $G \leq \text{SL}_2(\mathbb{C})$ be finite and $g \in G$. Let $\zeta$ be an eigenvalue of $g$. Then $\zeta^{-1}g$ is a reflection.

- So, if $G = \langle g_1, \ldots, g_r \rangle$, the group $\langle \zeta_1^{-1}g_1, \ldots, \zeta_r^{-1}g_r \rangle$ is an frg.

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Note that $\mathfrak{A}_5$, $\text{PSL}_2(7)$ and $3.\mathfrak{A}_6$ have faithful 3-dimensional representations and are generated by involutions.
Classification

The finite reflection groups on \( \mathbb{C} \) have been classified by Coxeter, Shephard and Todd. There is one infinite series \( G(\mathfrak{d}, \mathfrak{e}, \mathfrak{r}) \) (\( \mathfrak{d}, \mathfrak{e} \) and \( \mathfrak{r} \) integers), ... and 34 exceptional groups \( G_{4}, G_{5}, \ldots, G_{37} \).

The group \( G(\mathfrak{d}, \mathfrak{e}, \mathfrak{r}) \) (\( \mathfrak{d}, \mathfrak{e} \) and \( \mathfrak{r} \) integers) consists of all \( \mathfrak{r} \times \mathfrak{r} \) monomial matrices with entries in \( \mu \mathfrak{d} \) such that the product of entries belongs to \( \mu \mathfrak{d} \).

We have

\[
G(\mathfrak{d}, 1, \mathfrak{r}) \cong \mathbb{C}^{\mathfrak{d}} \rtimes S_{\mathfrak{r}}
\]

\[
G(2, 2, \mathfrak{r}) = \mathbb{W}(D_{\mathfrak{r}})
\]

\[
G_{23} = H_{3}, \quad G_{28} = F_{4}, \quad G_{30} = H_{4},
\]

\[
G_{35}, 36, 37 = E_{6}, 7, 8.
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Classification

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   \[ G(2,2,r) = W(D_r) \]
   \[ G_{23} = H_3, \quad G_{28} = F_4, \quad G_{30} = H_4 \]
   \[ G_{35,36,37} = E_{6,7,8}. \]
Let $G$ be a finite subgroup of $\text{GL}(V)$. A reflection $s$ is associated with

$H := \ker(s - 1)$, $L := \text{im}(s - 1)$, a reflecting pair $(H, L)$. Properties:

$H \oplus L = V$, $H$ determines $L$, and $L$ determines $H$, hence, in terms of normalizers, $\text{N}_G(H) = \text{N}_G(L) = \text{N}_G(H, L)$.

The fixator $G_H$ (pointwise stabilizer) of $H$ is a cyclic group consisting of reflections with reflecting hyperplane $H$ and reflecting line $L$.
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Reflection groups, braids, Hecke algebras
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Michel Broué  REFLECTION GROUPS, BRAIDS, HECKE ALGEBRAS
Notation

A := \{H \mid H \text{ reflecting hyperplane of some reflection in } G\}

For \(H \in A\), \(e_H := |G_H|\) is the generator of \(G_H\) whose nontrivial eigenvalue is \(e^{2\pi i / e_H}\), called a distinguished reflection.

For \(L\) a line in \(V\), the ideal \(q := SL\) of \(S\) is a height one prime ideal. In other words, the hypersurface of \(V\) defined by \(q\) is a codimension one irreducible variety.

Now the extension \(S_R = S \downarrow G \rtimes \uparrow \uparrow (\text{corresponding to the covering } V \downarrow V / G)\) is ramified at \(q = SL\) if and only if \(L\) is a reflecting line.
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Now the extension $R = S^G \hookrightarrow S$ (corresponding to the covering $\downarrow$) is ramified at $q = SL$ if and only if $L$ is a reflecting line.
Thus there are $G$-equivariant bijections $A \leftarrow \leftarrow \rightarrow \rightarrow \{\text{reflecting lines}\} \leftarrow \leftarrow \rightarrow \rightarrow \{\text{ramified height one prime ideals of } S\}$.

Assume $G$ generated by reflections.

1. The ramification locus of $V \rightarrow \rightarrow \rightarrow \rightarrow V/\!\!G$ is $\bigcup H \in A H$.

2. Let $X$ be a subset of $V$. Then the fixator of $X$ in $G$ is generated by the reflections which fix $X$.

3. The set $\text{Par}(G)$ of fixators ("parabolic subgroups" of $G$) is in (reverse–order) bijection with the set $I(A)$ of intersections of elements of $A$:

$$I(A) \sim_{\rightarrow} \text{Par}(G), X \mapsto G X.$$

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Reflection groups, braids, Hecke algebras
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$$I(\mathcal{A}) \sim \text{Par}(G), \quad X \mapsto G_X.$$
Braid groups

Set $V_{\text{reg}} := V - \bigcup_{H \in A} H$.

Since the covering $V_{\text{reg}} \to V_{\text{reg}}/G$ is Galois, it induces a short exact sequence

$$1 \to \Pi_1(V_{\text{reg}}, x_0) \to \Pi_1(V_{\text{reg}}/G, x_0) \to \mathbb{Z}^r \to 1.$$
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\( P_G \) (Pure braid group) \hspace{1cm} \( B_G \) (Braid group)
Notation around $H$

Let $H \in A$ with associated line $L$. For $x \in V$, we set $x = x_L + x_H$ (with $x_L \in L$ and $x_H \in H$).

Thus, we have $s_H(x) = e^{2\pi i/x_H x_L + x_H}$.

If $t \in \mathbb{R}$, we set $s_t H(x) = e^{2\pi i t/x_H x_L + x_H}$ defining a path $s_H$, $x$ from $x$ to $s_H(x)$.

We have $s_{te} H(x) = e^{2\pi i t/x_H x_L + x_H}$ defining a loop $\pi_H$, $x$ with origin $x$. In other words, $\pi_H$, $x = s_{se} H$, $x \in P$.
Notation around \( H \)

- Let \( H \in \mathcal{A} \), with associated line \( L \).
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- If $t \in \mathbb{R}$, we set:
  
  $$s^t_H(x) = e^{2i\pi t/e_H}x_L + x_H \quad \text{defining a path } s_{H,x} \text{ from } x \text{ to } s_H(x).$$
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- If $t \in \mathbb{R}$, we set:

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$$s_{H}^{te_H}(x) = e^{2\pi it}x_L + x_H \quad \text{defining a loop } \pi_{H,x} \text{ with origin } x.$$
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  $$s_{H}^{t}(x) = e^{2i\pi t/e_H} x_L + x_H \quad \text{defining a path } s_{H,x} \text{ from } x \text{ to } s_H(x).$$

  We have
  
  $$s_{H}^{te_H}(x) = e^{2\pi it} x_L + x_H \quad \text{defining a loop } \pi_{H,x} \text{ with origin } x.$$

  In other words,
  
  $$\pi_{H,x} = s_{H,x}^{e_H} \in P_G$$
Let $\gamma$ be a path in $V_{reg}$ from $x_0$ to $x_H$.

We define:

$$\sigma_{H,\gamma} := s_H(\gamma - 1) \cdot s_H, x_0 \cdot x_H \cdot s_H(x_H) \cdot x_0$$

**Definition**

We call braid reflections the elements $s_H, \gamma \in B$ defined by the paths $\sigma_{H,\gamma}$. 

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Reflection groups, braids, Hecke algebras
Braid reflections

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Braid reflections

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$$\sigma_H,\gamma = s_H(\gamma - 1) \cdot (s_H, x_0 \cdot \gamma, \gamma)$$

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Michel Broué
Reflection groups, braids, Hecke algebras
Braid reflections

Let $\gamma$ be a path in $V^{\text{reg}}$ from $x_0$ to $x_H$.

$$s_{H,x} \cdot \gamma$$

$\bullet$ $s_H(x_0)$

\[ \begin{array}{c}
H \\
\downarrow \\
x_H \\
\downarrow \\
\gamma \\
x_0 \\
\end{array} \]

$H$ \\
$s_H(x_H)$ \\
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- $s_{H,\gamma}$ and $s_{H,\gamma'}$ are conjugate in $P$.
- $s_{eH}^{H,\gamma}$ is a loop in $V^{\text{reg}}$.

\[ \gamma \]

\[ \cdot x_0 \]
The following properties are immediate.

- $s_{H,\gamma}$ and $s_{H,\gamma'}$ are conjugate in $P$.
- $s_{H,\gamma}^e$ is a loop in $V^{\text{reg}}$:

![Diagram](image)

The variety $V$ (resp. $V/G$) is connected, the hyperplanes are irreducible divisors (irreducible closed subvarieties of codimension one), and the braid reflections are “generators of the monodromy” around the irreducible divisors. Then
The following properties are immediate.

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\[
\begin{array}{c}
\circ \rightarrow \\
\gamma \cdot x_0
\end{array}
\]

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**Theorem**
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**Theorem**

1. The braid group \( B_G \) is generated by the braid reflections \( (s_{H,\gamma}) \) (for all \( H \) and all \( \gamma \)).
The following properties are immediate.

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**Theorem**

1. The braid group $B_G$ is generated by the braid reflections $(s_{H,\gamma})$ (for all $H$ and all $\gamma$).
2. The pure braid group $P_G$ is generated by the elements $(s_{H,\gamma}^{eH})$.
Linear characters of the reflection groups

For $H \in \mathcal{A}$,
Linear characters of the reflection groups

For $H \in \mathcal{A}$,

- $j_H$ denotes a nontrivial element of $L$, 

Proposition 1

The linear character $\det_H : G \to \mathbb{C} \times$ is defined by $g(j_H) = \det_H(g)j_H$. 

$\det_H(s) = \begin{cases} 
\det(s) & \text{if } Hs = G \\
1 & \text{if not}
\end{cases}$

$\hom(G, \mathbb{C} \times) \sim \rightarrow (\prod_{H \in \mathcal{A}} \hom(G_H, \mathbb{C} \times)) / G 
\cong \prod_{H \in \mathcal{A}} \hom(G_H, \mathbb{C} \times)$
Linear characters of the reflection groups

For $H \in \mathcal{A}$,

- $j_H$ denotes a nontrivial element of $L$,
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3. $\text{Hom}(G, \mathbb{C}^\times) \xrightarrow{\sim} (\prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times))^{G}$
Linear characters of the reflection groups

For $H \in A$,

- $j_H$ denotes a nontrivial element of $L$,

- $j_H := \prod \{ H' \mid (H' =_G H) \} j_{H'}$ (depends only on the orbit of $H$ under $G$)

**Proposition**

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3. $\text{Hom}(G, \mathbb{C}^\times) \overset{\sim}{\longrightarrow} \left( \prod_{H \in A} \text{Hom}(G_H, \mathbb{C}^\times) \right)^G \cong \prod_{H \in A/G} \text{Hom}(G_H, \mathbb{C}^\times)$
Linear characters of the braid groups

The discriminant at $H \in A$ (or rather $A/G$) is $\Delta_H := j_H H \Delta_H \in R = S_G$ hence its dual $\Delta^*_H \in R^* = S_G^*$ defines a (continuous) map $\Delta^*_H : V_{\text{reg}} \to C \times V_{\text{reg}} / G$ hence defines a morphism $\Pi_1(\Delta^*_H) : \Pi_1(V_{\text{reg}} / G) \to \Pi_1(C \times)$ i.e., $\ell_H : B_G \to \mathbb{Z}$ For $H \in A_L$, $L_H \downarrow \downarrow \downarrow \downarrow \downarrow B_G \cong \mathbb{Z} \downarrow G \cong \mathbb{Z} / e_H \mathbb{Z}$
Linear characters of the braid groups

- The discriminant at $H \in \mathcal{A}$ (or rather $\mathcal{A}/G$) is $\Delta_H := j_H^{\text{et}}$
Linear characters of the braid groups

- The discriminant at \( H \in A \) (or rather \( A/G \)) is \( \Delta_H := j_H^{eH} \)
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Linear characters of the braid groups

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- $\Delta_H \in R = S^G$ hence its dual $\Delta_H^* \in R^* = S^*_G$ defines a (continuous) map

$$\Delta_H^* : V^{\text{reg}} \rightarrow \mathbb{C}^\times \rightarrow V^{\text{reg}}/G$$
Linear characters of the braid groups

- The discriminant at \( H \in \mathcal{A} \) (or rather \( \mathcal{A}/G \)) is \( \Delta_H := j_{eH}^H \)

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\[
V^{\text{reg}} / G
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$$\Pi_1(\Delta_H^*) : \Pi_1(V^\text{reg}/G) \to \Pi_1(\mathbb{C}^\times) \quad i.e.,$$
The discriminant at $H \in \mathcal{A}$ (or rather $\mathcal{A}/G$) is $\Delta_H := j_H^e$

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For \( H \in \mathcal{A} \),

\[
\text{Michel Broué}
\]

Reflection groups, braids, Hecke algebras
Linear characters of the braid groups

- The discriminant at $H \in \mathcal{A}$ (or rather $\mathcal{A}/G$) is $\Delta_H := j^e_H$

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$$

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- For $H \in \mathcal{A}$,

$$
B_{GH} \simeq \mathbb{Z}
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The discriminant at \( H \in \mathcal{A} \) (or rather \( \mathcal{A}/G \)) is \( \Delta_H := j_H^{e_H} \).

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\[ \Delta_H^*: V^{\text{reg}} \rightarrow \mathbb{C}^\times \]

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For \( H \in \mathcal{A} \),

\[ B_{G_H} \simeq \mathbb{Z} \]

\[ G_H \simeq \mathbb{Z}/e_H\mathbb{Z} \]
Proposition

\[ \text{Hom}(G, C \times \prod_{H \in A} \text{Hom}(G H, C)) \sim - \rightarrow \text{Hom}(B G, Z) \sim - \rightarrow \prod_{H \in A} \text{Hom}(B G H, Z) \]

\[ \ell_H(\text{sn}_1 H_1 \cdot \text{sn}_2 H_2 \cdot \cdots \cdot \text{sn}_k H_k, \gamma_1 \cdot \gamma_2 \cdots \gamma_k) = \sum_{\{i \mid H_i = G H\}} n_i \]

Michel Broué

Reflection groups, braids, Hecke algebras
Proposition
Proposition

1. \( \text{Hom}(G, \mathbb{C}^\times) \xrightarrow{\approx} (\prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times))^G \)
Proposition

1. \[ \text{Hom}(G, \mathbb{C}^\times) \sim (\prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times))^G \]

\[ \text{Hom}(B_G, \mathbb{Z}) \sim (\prod_{H \in \mathcal{A}} \text{Hom}(B_{G_H}, \mathbb{Z}))^G \]
Proposition

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2. $\ell_H$ is a length:
Proposition

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\( \ell_H \) is a length:

\[
\ell_H(s_{H_1, \gamma_1}^{n_1} \cdot s_{H_2, \gamma_2}^{n_2} \cdots s_{H_k, \gamma_k}^{n_k}) = \sum_{\{i \mid (H_i = G)\}} n_i
\]
Proposition

1. \( \text{Hom}(G, \mathbb{C}^\times) \overset{\sim}{\to} (\prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times))^G \)

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\]
Center of the braid groups

From now on we assume that $G$ is irreducible on $V$. Hence the centre of $G$ is cyclic. Set $z := |\mathbb{Z}_G|$ and $\zeta := e^{2\pi i/z}$.

Let $\pi \in P_G$ defined by $\pi : t \mapsto e^{2\pi it}x_0$.

Let $\zeta \in B_G$ defined by $\zeta : t \mapsto e^{2\pi it/z}x_0$.

Theorem 1

$Z_{P_G} = \langle \pi \rangle$ and $Z_{B_G} = \langle \zeta \rangle$.

We have the short exact sequence

$1 \rightarrow Z_{P_G} \rightarrow Z_{B_G} \rightarrow Z_G \rightarrow 1$
Center of the braid groups

From now on we assume that $G$ is irreducible on $V$. 

Theorem

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2

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**Theorem**

1. $ZP_G = \langle π \rangle$ and $ZB_G = \langle ζ \rangle$. 
From now on we assume that $G$ is irreducible on $V$. Hence the centre of $G$ is cyclic. Set $z := |ZG|$ and $\zeta := e^{2i\pi/z}$.

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Theorem

1. $ZP_G = \langle \pi \rangle$ and $ZB_G = \langle \zeta \rangle$.
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$$1 \longrightarrow ZP_G \longrightarrow ZB_G \longrightarrow ZG \longrightarrow 1$$
Case of Coxeter groups

The choice of a Coxeter generating set for $\mathcal{G}$ defines a presentation of $B\mathcal{G}$.

Example:

\[ \begin{align*}
\pi &= ((st)^{t_1}t_2\cdots t_{r-1})^2
\end{align*} \]

Let $w_0$ be the longest element of $\mathcal{G}$, and let $g_0$ be its lift in $B\mathcal{G}$.

\[ \pi = g_2^2 \]

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The choice of a Coxeter generating set for $G$ defines a presentation of $B_G$.
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Case of Coxeter groups

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Example:

\[
\begin{array}{cccc}
s & t_1 & t_2 & t_{r-1} \\
\end{array}
\]

and a “section” (not a group morphism !) of the map $B_G \to G$ using reduced decompositions.
Case of Coxeter groups

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Case of Coxeter groups

The choice of a Coxeter generating set for $G$ defines a presentation of $B_G$

\[
\begin{array}{c}
\text{Example:} \\
\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \\
\scriptsize s \quad t_1 \quad t_2 \quad \cdots \quad t_{r-1}
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Let $w_0$ be the longest element of $G$, and let $g_0$ be its lift in $B_G$.

$$\pi = g_0^2$$

Example: $$\pi = (st_1t_2 \cdots t_{r-1})^{2r}$$
Artin–like presentations

An Artin–like presentation is \( \langle s \in S \mid \{ v_i = w_i \} \rangle \)
where \( S \) is a finite set of distinguished braid reflections, \( I \) is a finite set of relations which are multi–homogeneous.

Theorem (Bessis)

Let \( G \subset \text{GL}(V) \) be a complex reflection group. Let \( d_1 \leq d_2 \leq \cdots \leq d_r \) be the family of its invariant degrees.

1. The following integers are equal (denoted by \( \Gamma_G \)):
   - The minimal number of reflections needed to generate \( G \)
   - The minimal number of braid reflections needed to generate \( B_G \)
   \[ \left\lceil \frac{N + N_h}{d_r} \right\rceil \]

2. Either \( \Gamma_G = r \) or \( \Gamma_G = r + 1 \), and the group \( B_G \) has an Artin–like presentation by \( \Gamma_G \) braid reflections.

Michel Broué
Reflection groups, braids, Hecke algebras
Artin–like presentations

An Artin–like presentation is

$$\langle s \in S \mid \{v_i = w_i\}_{i \in I} \rangle$$

where

Theorem (Bessis)

Let $G \subset \text{GL}(V)$ be a complex reflection group. Let $d_1 \leq d_2 \leq \cdots \leq d_r$ be the family of its invariant degrees.

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   - The minimal number of reflections needed to generate $G$
   - The minimal number of braid reflections needed to generate $B_G$
   $$\left\lceil \frac{N + N_h}{d_r} \right\rceil$$

2. Either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group $B_G$ has an Artin–like presentation by $\Gamma_G$ braid reflections.

Michel Broué

Reflection groups, braids, Hecke algebras
Artin–like presentations

An Artin–like presentation is

$$\langle s \in S \mid \{v_i = w_i\}_{i \in I} \rangle$$

where

- $S$ is a finite set of distinguished braid reflections,
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- \( S \) is a finite set of distinguished braid reflections,
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  _i.e._, such that (for each $i$) $v_i$ and $w_i$ are positive words in elements of $S$
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Theorem (Bessis)

Let $G \subset GL(V)$ be a complex reflection group. Let $d_1 \leq d_2 \leq \cdots \leq d_r$ be the family of its invariant degrees.

$$\text{either } \Gamma_G = r \text{ or } \Gamma_G = r + 1, \text{ and the group } B_G \text{ has an Artin–like presentation by } \Gamma_G \text{ braid reflections.}$$
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Reflection groups, braids, Hecke algebras
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**Theorem (Bessis)**

Let $G \subset \text{GL}(V)$ be a complex reflection group. Let $d_1 \leq d_2 \leq \cdots \leq d_r$ be the family of its invariant degrees.

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   - The minimal number of reflections needed to generate $G$
   - The minimal number of braid reflections needed to generate $B_G$
   - $\lceil (N + N_h)/d_r \rceil$ ($N := \text{number of reflections}, N_h := \text{number of hyperplanes}$)
Artin–like presentations

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2. Either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group $B_G$ has an Artin–like presentation by $\Gamma_G$ braid reflections.
Let $D$ be a diagram like $s \circ a \circ b \circ c$. $D$ represents the relations $s \cdot t \cdot s \cdot u \cdot s \cdot t \cdot s$, $e$ factors $= t \cdot u \cdot s \cdot t \cdot s \cdot t \cdot s$, $e$ factors $= u \cdot t \cdot s \cdot u \cdot t \cdot s$. We denote by $D_{br}$ and call braid diagram the diagram $s \circ n \circ e \circ t \circ u$, which represents the relations $s \cdot t \cdot s \cdot u \cdot s \cdot t \cdot s$, $e$ factors $= t \cdot u \cdot s \cdot t \cdot s \cdot t \cdot s$, $e$ factors $= u \cdot t \cdot s \cdot u \cdot t \cdot s$. Note that $G_{72}$: $s \circ 3 \circ n \circ 3 \circ t \circ 3 \circ u$, $G_{11}$: $s \circ 2 \circ n \circ 3 \circ t \circ 4 \circ u$, $G_{19}$: $s \circ 2 \circ n \circ 3 \circ t \circ 5 \circ u$ have same braid diagram.
Let $D$ be a diagram like

$$s \quad e \quad b \quad t \quad \quad \quad c \quad u$$

Note that $G_7$ has the same braid diagram.
Let $\mathcal{D}$ be a diagram like

\[ s \quad a \quad e \quad b \quad t \quad c \quad u \]

$\mathcal{D}$ represents the relations

\[ stustu \cdots = tustus \cdots = ustust \cdots \]

$e$ factors $e$ factors $e$ factors

Note that $G_7$, $G_{11}$, and $G_{19}$ have the same braid diagram.
Let $\mathcal{D}$ be a diagram like

$$s \circ e \circ t \circ u$$

$\mathcal{D}$ represents the relations

$$stustu \cdots = tustus \cdots = ustust \cdots$$

e factors e factors e factors

and

$$s^a = t^b = u^c = 1$$
The braid diagrams

Let $\mathcal{D}$ be a diagram like

\[ s \begin{array}{c} \circ \\ a \end{array} e \begin{array}{c} \circ \\ b \end{array} t \begin{array}{c} \circ \\ c \end{array} u \]

$\mathcal{D}$ represents the relations

\[ stustu \cdots = tustus \cdots = ustust \cdots \]

and

\[ s^a = t^b = u^c = 1 \]

We denote by $\mathcal{D}_{br}$ and call \textit{braid diagram} the diagram

\[ s \begin{array}{c} \circ \\ \end{array} e \begin{array}{c} \circ \\ \end{array} u \begin{array}{c} \circ \\ t \end{array} \]
The braid diagrams

Let $\mathcal{D}$ be a diagram like

\[ s \quad e \quad t \quad u \]

\( s \circ \quad e \quad t \quad u \)

$\mathcal{D}$ represents the relations

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and

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\[ s \quad e \quad u \]

which represents the relations

\[ stustu \cdots = tustus \cdots = ustust \cdots \]
The braid diagrams

Let $\mathcal{D}$ be a diagram like $s \circ a \circ e \circ b \circ t \circ c \circ u$. $\mathcal{D}$ represents the relations

\[
\underbrace{stustu\cdots} = \underbrace{tustus\cdots} = \underbrace{ustust\cdots}
\]

e factors e factors e factors

and $s^a = t^b = u^c = 1$

We denote by $\mathcal{D}_{br}$ and call braids diagram the diagram which represents the relations

\[
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e factors e factors e factors

Note that

$G_7 : s \circ 2 \circ 3 \circ t \circ 3 \circ u$

$G_{11} : s \circ 2 \circ 3 \circ t \circ 4 \circ u$

$G_{19} : s \circ 2 \circ 3 \circ t \circ 5 \circ u$
The braid diagrams

Let $\mathcal{D}$ be a diagram like

\[ s \circ e \circ c \circ u \]

$\mathcal{D}$ represents the relations

\[
\begin{align*}
stustu \cdots &= tustus \cdots = ustust \cdots \\
\text{e factors} &\quad \text{e factors} &\quad \text{e factors}
\end{align*}
\]

and

\[ s^a = t^b = u^c = 1 \]

We denote by $\mathcal{D}_{br}$ and call braid diagram the diagram

\[ s \circ e \circ u \]

which represents the relations

\[
\begin{align*}
stustu \cdots &= tustus \cdots = ustust \cdots \\
\text{e factors} &\quad \text{e factors} &\quad \text{e factors}
\end{align*}
\]

Note that

\[
\begin{align*}
G_7 : s \circ t \circ u \\
G_{11} : s \circ t \circ u \\
G_{19} : s \circ t \circ u
\end{align*}
\]

have same braid diagram.
For each irreducible complex irreducible group $G$, there is a diagram $D$, whose set of nodes $\mathcal{N}(D)$ is identified with a set of distinguished reflections in $G$, such that

Theorem

For each $s \in \mathcal{N}(D)$, there exists a braid reflection $s \in B_G$ above $s$ such that the set

\[
\{s\}_{s \in \mathcal{N}(D)},
\]

together with the braid relations of $D_{br}$, is a presentation of $B_G$.

The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\bullet_d e \bullet_d$, corresponding to the presentation $s_d = t_d = 1$ and $s t s t s t \cdots \approx \epsilon$ factors $= t s t s t \cdots \approx \epsilon$ factors.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$,
For each irreducible complex irreducible group $G$, there is a diagram $D$, whose set of nodes $\mathcal{N}(D)$ is identified with a set of distinguished reflections in $G$. Theorem: For each $s \in \mathcal{N}(D)$, there exists a braid reflection $s \in B_G$ above $s$ such that the set \{s\} $s \in \mathcal{N}(D)$, together with the braid relations of $D_{br}$, is a presentation of $B_G$. The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $s_d e \circ d_t$ corresponding to the presentation $s_d = t_d = 1$ and factors $s_e = t_e$. Michel Broué
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

\begin{equation}
\text{Theorem}
\end{equation}

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set 
\begin{equation}
\{s\} \cup \{s \in \mathcal{N}(\mathcal{D})\}
\end{equation}

together with the braid relations of $\mathcal{D}$, is a presentation of $B_G$.

The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type \(\ast\), corresponding to the presentation $s = t = 1$ and $ststs \cdots = tsts \cdots$.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$. 
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- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\begin{array}{c}
  \circlearrowleft \\
  \circ \\
  \circlearrowright
\end{array}$.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$.

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\circled{d} \\
\circled{e} \\
\circled{d}
\end{array}$

  , corresponding to the presentation
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

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- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\overset{s}{\circ} \overset{e}{\circ} \overset{t}{\circ}$, corresponding to the presentation

\[
s^d = t^d = 1
\]
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- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\begin{tikzpicture} \node[asymmetry] (d) at (0,0) {d}; \node[asymmetry] (e) at (1,0) {e}; \node[asymmetry] (d) at (2,0) {d}; \node[asymmetry] (s) at (0,0) {s}; \node[asymmetry] (t) at (2,0) {t}; \draw (s) edge (d) (d) edge (e) (e) edge (d) (d) edge (t); \end{tikzpicture}$, corresponding to the presentation

\[ s^d = t^d = 1 \text{ and } \underbrace{ststs \cdots}_{\text{e factors}} = \underbrace{tstst \cdots}_{\text{e factors}} \]
The group $G_{18}$ has diagrams corresponding to the presentation $s_5 t_3 = 1$ and $stst = tst$.

The group $G_{31}$ has diagrams corresponding to the presentation $s_2 t_2 = 1$, $uv = vu$, $sw = ws$, $vw = wv$, $sut = uts = tsu$, $svs = vsv$, $tv t = vtv$, $t wt = wt w$, $uw u = uw u$. 

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The group $G_{18}$ has diagram \[
\begin{array}{c}
\circ & \circ & \circ \\
 & s & t \\
\end{array}
\] corresponding to the presentation
\[
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\]
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The group $G_{18}$ has diagram $\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
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The group $G_{31}$ has diagram $\begin{array}{c}
\circ \quad \circ \quad \circ \\
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\end{array}$ corresponding to the presentation

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The group $G_{18}$ has diagram corresponding to the presentation

\[ s^5 = t^3 = 1 \text{ and } stst = tsts. \]

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\[
\begin{align*}
    s^2 &= t^2 = u^2 = v^2 = w^2 = 1, \\
    uv &= vu, \quad sw = ws, \quad vw = vw, \quad sut &= uts = tsu,
\end{align*}
\]
• The group $G_{18}$ has diagram $\begin{array}{c} 5 \\ \hline s \\ \hline 3 \\ \hline t \end{array}$ corresponding to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts.$$ 

• The group $G_{31}$ has diagram $\begin{array}{c} 2 \\ \hline s \\ \hline 2 \\ \hline t \\ \hline 2 \\ \hline w \end{array}$ corresponding to the presentation

$$s^2 = t^2 = u^2 = v^2 = w^2 = 1,$$

$$uv = vu, \ \text{sw} = ws, \ \text{vw} = wv, \ \text{sut} = uts = tsu,$$

$$svs = vsv, \ tvt = vtv, \ twt = wtw, \ wuw = uwu.$$
More on the work of Bessis

• Solution of an old conjecture

Theorem

The space $V_{\text{reg}}$ is a $K(\pi,1)$.

• Springer's theory of regular elements

An element $g \in G$ is $\zeta$-regular if $g$ has a regular eigenvector for the eigenvalue $\zeta$.

Denote by $V(g,\zeta)$ the $\zeta$-eigenspace of $g$ in $V$.

Theorem (Springer, 1974)

Let $g \in G$ be $d$-regular. Then $C_G(g)$ is a frg on $V(g,\zeta)$, with set of degrees $\{d_i | d_i \text{ divides } d\}$.

Idea of proof: show that $C_G(g)$ has polynomial invariants on $V(g,\zeta)$. 

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Examples

\[ W = S \text{in its natural permutation representation on } V = C_n. \]

Assume that \( d \mid r \).

Then the product of \( r/d \) disjoint \( d \)-cycles is \( d \)-regular, with centralizer \( C_d \wr S_r/d \), with degrees \( \{ d^i | d \text{ divides } d^i \} = \{ d, 2d, \ldots, d \cdot r/d = r \} \).

\[ G = W(F_4), \text{ a Weyl group. There exist } 3\text{-regular elements in } G. \]

The degrees of \( W(F_4) \) are 2, 6, 8, 12, so the centralizer has degrees 6, 12: It is the complex reflection group \( G_5 \).

So, even if \( G \) is a Weyl group, \( C_G \) may be a truly complex reflection group.
Examples

- \( \mathcal{W} = \mathfrak{S}_r \) in its natural permutation representation on \( V = \mathbb{C}^n \). Assume that \( d \mid r \).
Examples

- $W = \mathfrak{S}_r$ in its natural permutation representation on $V = \mathbb{C}^n$. Assume that $d|r$.
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\[
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\]
Examples

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\[
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\]

• \( G = W(F_4) \), a Weyl group. There exist 3-regular elements in \( G \).
Examples

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Examples

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Examples

- \( W = \mathfrak{S}_r \) in its natural permutation representation on \( V = \mathbb{C}^n \).
  Assume that \( d | r \).
  Then the product of \( r/d \) disjoint \( d \)-cycles is \( d \)-regular, with centralizer \( C_d \wr \mathfrak{S}_{r/d} \), with degrees

  \[
  \{ d_i \mid d \text{ divides } d_i \} = \{ d, 2d, \ldots, d \cdot r/d = r \}.
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- \( G = W(F_4) \), a Weyl group. There exist 3-regular elements in \( G \). The degrees of \( W(F_4) \) are 2, 6, 8, 12, so the centralizer has degrees 6, 12:
  It is the complex reflection group \( G_5 \).

So, even if \( G \) is a Weyl group, \( C_G(g) \) may be a truly complex reflection group.
Springer’s theory of regular elements in complex reflections groups lifts to braid groups

Theorem (Bessis, 2007)

\[ \zeta_d = e^{2i\pi/d}. \]

The \( \zeta_d \)-regular elements in \( G \) are the images of the \( d \)-th roots of \( \pi \).

All \( d \)-th roots of \( \pi \) are conjugate in \( B_G \).

Let \( g \) be a \( d \)-th root of \( \pi \), with image \( g \) in \( G \). Then \( C_{B_G}(g) \) is the braid group of \( C_G(g) \).
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A monodromy representation

(after Knizhnik–Zamolodchikov, Cherednik, Dunkl, Opdam, Kohno, Broué-Malle-Rouquier)

For $H \in A$, let $\alpha_H$ be a linear form with kernel $H$, and $\omega_H := \frac{1}{2} i \pi d \alpha_H \alpha_H$. Each family $(z_H)_H \in A \in \prod_{H \in A} \mathbb{C}G_H$ defines a $G$-invariant differential form on $V_{reg}$ with values in $\mathbb{C}G$, hence a linear differential equation $df = \omega f$ for $f : V_{reg} \rightarrow \mathbb{C}G$, i.e., $\forall v \in V, x \in V_{reg}$, $df(x)(v) = \frac{1}{2} i \pi \sum_{H \in A} \alpha_H(v) \alpha_H(x) z_H f(x)$.
For $H \in \mathcal{A}$, let $\alpha_H$ be a linear form with kernel $H$, 

Each family $(z_H)_{H \in \mathcal{A}}$ defines a $G$-invariant differential form on $V_{\text{reg}}$ with values in $C^G$.

Hence a linear differential equation $df = \omega f$ for $f : V_{\text{reg}} \to C^G$, i.e., 

$$
\forall v \in V, \ x \in V_{\text{reg}},
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For $H \in \mathcal{A}$, let $\alpha_H$ be a linear form with kernel $H$, and

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For $H \in \mathcal{A}$, \{ 

- $G \vee H$ is the group of characters of $G_H$,
- for $\theta \in G \vee H$, $e_{H,\theta}$ is the corresponding primitive idempotent in $C_{G_H}$.

We set $q_H := \exp \left( -2i\pi/e_H \right) z_H =: \sum_{\theta \in G \vee H} q_{H,\theta} e_{H,\theta}$.

Theorem 1

The form $\omega$ is integrable, hence defines a group morphism $\rho : B_G \to (C_{G_H}) \times$.

2

Whenever $s_H, \gamma$ is a braid reflection around $H$, there is $u_H \in (C_{G_H}) \times$ such that $\rho(s_H, \gamma) = u_H(q_{H,s_H})u_H^{-1}$. In particular, we have $\prod_{\theta \in G \vee H} (\rho(s_H, \gamma) - q_{H,\theta}) = 0$. 
For $H \in \mathcal{A}$, 

\begin{align*}
\begin{cases}
\cdot G_H^\vee & \text{is the group of characters of } G_H,
\end{cases}
\end{align*}
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\begin{equation*}
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\end{cases}
\end{align*}
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Theorem
For $H \in \mathcal{A}$, \begin{itemize}
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1. The form $\omega$ is integrable, hence defines a group morphism $\rho : B_G \longrightarrow (\mathbb{C}G)^\times$.

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In particular, we have 
\[
\prod_{\theta \in G_H^\vee} (\rho(s_{H, \gamma}) - q_{H, \theta}\theta(s_H)) = 0.
\]
Hecke algebras

Every complex reflection group $G$ has an Artin-like presentation:

$G_2$: $s_2$, $t_2$,

$G_4$: $s_3$, $t_3$,

and a field of realization $Q_G := Q(\{\text{tr} V(g) | g \in G\})$.

The associated generic Hecke algebra is defined from such a presentation:

$H(G_2) := <S, T; \begin{cases} STSTST = TSTSTS \quad (S - q_0)(S - q_1) = 0 \\ (T - r_0)(T - r_1) = 0 \end{cases}>$

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Every complex reflection group $G$ has an Artin-like presentation:

$G_2 : \begin{array}{l} 2 \\ s \end{array} \begin{array}{l} 2 \\ t \end{array}$, \quad $G_4 : \begin{array}{l} 3 \\ s \end{array} \begin{array}{l} 3 \\ t \end{array}$
Every complex reflection group $G$ has an Artin-like presentation:

\[ G_2 : \begin{array}{cc}
& 2 \\
\_ & \_ \\
s & t
\end{array} , \quad G_4 : \begin{array}{cc}
& 3 \\
\_ & \_ \\
s & t
\end{array} \]

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Every complex reflection group $G$ has an Artin-like presentation:

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Theorem (G. Malle and al.)

1. The generic Hecke algebra $\mathcal{H}(G)$ is free of rank $|G|$ over the corresponding Laurent polynomial ring $\mathbb{Z}[(q_i^\pm 1), (r_j^\pm 1), \ldots]$.
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2. It becomes a **split semisimple algebra** over a field obtained by extracting suitable roots of the indeterminates:

   - Through the specialisation $x_i \mapsto 1$, $y_j \mapsto 1$, ..., that algebra becomes the group algebra of $G$ over $\mathbb{Q}G$.
   - The above specialisation defines a bijection $\text{Irr}(G) \xrightarrow{\sim} \text{Irr}(\mathcal{H}(G))$, $\chi \mapsto \chi_{\mathcal{H}}$.
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   \text{if } G = \overset{d}{\underset{s}{\square}} \overset{m}{\underset{t}{\circ}} \cdots, \text{ then for } \]

   \[
   (x_i^{\mu(\mathbb{Q}_G)}) = \zeta_d^{-i} q_i \quad i = 0, 1, \ldots, d - 1 \quad ; \quad (y_j^{\mu(\mathbb{Q}_G)}) = \zeta_e^{-j} r_j \quad j = 0, 1, \ldots, e - 1
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Michel Broué  \hspace{2.5em} Reflection groups, braids, Hecke algebras
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- Through the specialisation $x_i \mapsto 1, y_j \mapsto 1, \ldots$, that algebra becomes the group algebra of $G$ over $\mathbb{Q}_G$.
- The above specialisation defines a bijection

   \[
   \text{Irr}(G) \overset{\sim}{\rightarrow} \text{Irr}(\mathcal{H}(G)), \quad \chi \mapsto \chi_{\mathcal{H}}.
   \]
There exists a unique linear form $t_q: H(W, q) \rightarrow \mathbb{Z}[q, q^{-1}]$ with the following properties.

1. $t_q$ is a symmetrizing form on the algebra $H(W, q)$.
2. $t_q$ specializes to the canonical linear form on the group algebra.
3. For all $b \in B$, we have $t_q(b \pi) = t_q(b) \pi$.
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Theorem–Conjecture

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- \( t_q \) is a symmetrizing form on the algebra \( \mathcal{H}(W, q) \).
- \( t_q \) specializes to the canonical linear form on the group algebra.
- For all \( b \in B \), we have

\[ t_q(b^{-1})^\vee = \frac{t_q(b\pi)}{t_q(\pi)}. \]
The form $t_q$ satisfies the following conditions. As an element of $\mathbb{Z}[q, q^{-1}]$, $t_q(b)$ is multi–homogeneous with degree $\ell H(b)$ in the indeterminates $q_H, \theta$.

If $W'$ is a parabolic subgroup of $W$, the restriction of $t_q$ to a parabolic sub–algebra $H(W', W, q)$ is the corresponding specialization of $t_q'(W')$.

The canonical forms $t_q$ are hidden behind Lusztig's theory of characters of finite reductive groups, their generic degrees and Fourier transform matrices.
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The canonical forms $t_q$ are hidden behind Lusztig’s theory of characters of finite reductive groups, their generic degrees and Fourier transform matrices.
Cyclotomic algebras

Let $\zeta$ be a root of unity. A $\zeta$–cyclotomic specialisation of the generic Hecke algebra is a morphism

$$\varphi : x_i \mapsto (\zeta^{-1}q)^{m_i}, \ y_j \mapsto (\zeta^{-1}q)^{n_j}, \ldots \ (m_i, n_j \in \mathbb{Z}),$$

which gives rise to a $\zeta$–cyclotomic Hecke algebra $\mathcal{H}_\varphi(G)$. 

Michel Broué
Reflection groups, braids, Hecke algebras
Cyclotomic algebras

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A 1–cyclotomic Hecke algebra for \( G_2 = \begin{array}{c} \text{2} \\ \hline \text{s} \end{array} \cong \begin{array}{c} \text{2} \\ \hline \text{t} \end{array} \):}

\[
< S, T ; \begin{cases}
STSTST = TSTSTS \\
(S - q^2)(S + 1) = 0 \\
(T - q)(T + 1) = 0
\end{cases} >
\]
A $\zeta_3$–cyclo-tomic Hecke algebra for $B_2(3) = \begin{array}{c} 3 \\ s \end{array} \begin{array}{c} 2 \\ t \end{array}$:

$$< S, T ; \begin{cases} STST = TSTS \\ (S - 1)(S - q)(S - q^2) = 0 \\ (T - q^3)(T + 1) = 0 \end{cases} >$$
A $\zeta_3$–cycloptomic Hecke algebra for $B_2(3) = \begin{array}{c} 3 \\ s \end{array} \begin{array}{c} 2 \\ t \end{array}$:

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Relevance to character theory of finite reductive groups

The unipotent characters in a given $d$–Harish–Chandra series $\text{UnCh}(G^F; (L, \lambda))$ are described by a suitable $\zeta_d$–cycloptomic Hecke algebra $\mathcal{H}_{G^F}(L, \lambda)$ for the corresponding $d$–cycloptomic Weyl group $W_{G^F}(L, \lambda)$:

$$\text{UnCh}(G^F; (L, \lambda)) \leftrightarrow \text{Irr}(\mathcal{H}_{G^F}(L, \lambda)).$$