Kazhdan-Lusztig theory with unequal parameters (Part III)

Cédric Bonnafé

CNRS - Université de Franche-Comté (Besançon)

Cambridge, January 2009

The decomposition into left, right and two-sided cells is known in the following cases:

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...
- Unequal parameter case:
  - *Quasi-split* case (Lusztig)

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...
- Unequal parameter case:
  - *Quasi-split* case (Lusztig)
  - Type B_n, $b > (n-1)a$ (Iancu-B., 2003)

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...
- Unequal parameter case:
  - *Quasi-split* case (Lusztig)
  - Type B_n, $b > (n-1)a$ (Iancu-B., 2003)
  - Type F_4 (Geck, 2004)

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...
- Unequal parameter case:
  - *Quasi-split* case (Lusztig)
  - Type B_n, $b > (n-1)a$ (Iancu-B., 2003)
  - Type F_4 (Geck, 2004)
  - Type $\tilde{G}_2$, $\tilde{B}_2$ (**ALL** choices of parameters, Guilhot, 2009)

The decomposition into left, right and two-sided cells is known in the following cases:

- Equal parameter case:
  - Type A (Kazhdan-Lusztig 1979)
  - Type $\tilde{B}_2$, $\tilde{G}_2$ (Lusztig, 1986)
  - Type B, D (Garfinkle, Barbasch-Vogan, '90)
  - Type $\tilde{A}$
  - ...
- Unequal parameter case:
  - *Quasi-split* case (Lusztig)
  - Type B_n, $b > (n-1)a$ (Iancu-B., 2003)
  - Type F_4 (Geck, 2004)
  - Type $\tilde{G}_2$, $\tilde{B}_2$ (**ALL** choices of parameters, Guilhot, 2009)
  - ...

Left-connectedness

Left-connectedness

A subset $\mathscr{E}$ of W is called left-connected if, for all $x, y \in W$, there exists a sequence $s_1, \ldots, s_r \in S$ such that $y = s_r \cdots s_2 s_1 x$ and $s_i \cdots s_2 s_1 x \in \mathscr{E}$ for all i.

Left-connectedness

A subset $\mathscr{E}$ of W is called left-connected if, for all $x, y \in W$, there exists a sequence $s_1, \ldots, s_r \in S$ such that $y = s_r \cdots s_2 s_1 x$ and $s_i \cdots s_2 s_1 x \in \mathscr{E}$ for all i.

Conjecture (Lusztig).

Every left cell is left-connected.

Left-connectedness

A subset $\mathscr{E}$ of W is called left-connected if, for all $x, y \in W$, there exists a sequence $s_1, \ldots, s_r \in S$ such that $y = s_r \cdots s_2 s_1 x$ and $s_i \cdots s_2 s_1 x \in \mathscr{E}$ for all i.

Conjecture (Lusztig).

Every left cell is left-connected.
Left cells are left-connected components of two-sided cells.

Semi-continuity properties

Semi-continuity properties

Assume here that $\Gamma = \mathbb{R}$. Let V be the vector space of maps $\varphi : S/\sim \longrightarrow \mathbb{R}_{>0}$ and let $V^{\#}$ be the subset of maps $\varphi : S/\sim \longrightarrow \mathbb{R}_{>0}$.

Conjecture (B.)

There exists a finite set of (linear) rational hyperplanes $\mathscr{A}$ in V such that:

Semi-continuity properties

Assume here that $\Gamma = \mathbb{R}$. Let V be the vector space of maps $\varphi : S/{\sim} \longrightarrow \mathbb{R}_{>0}$ and let $V^{\#}$ be the subset of maps $\varphi : S/{\sim} \longrightarrow \mathbb{R}_{>0}$.

Conjecture (B.)

There exists a finite set of (linear) rational hyperplanes $\mathscr{A}$ in V such that:

- If φ and φ' belong to the same $\mathscr{A}$-facet in $V^{\#}$, then the left (right, two-sided) cells for (W, S, φ) and (W, S, φ') coincide.

Semi-continuity properties

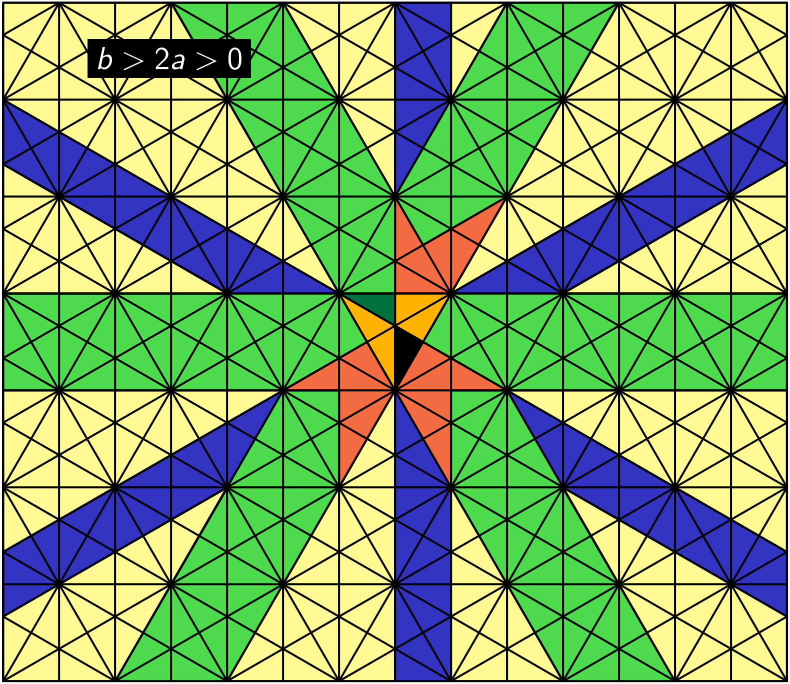
Assume here that $\Gamma = \mathbb{R}$. Let V be the vector space of maps $\varphi : S/{\sim} \longrightarrow \mathbb{R}_{>0}$ and let $V^{\#}$ be the subset of maps $\varphi : S/{\sim} \longrightarrow \mathbb{R}_{>0}$.

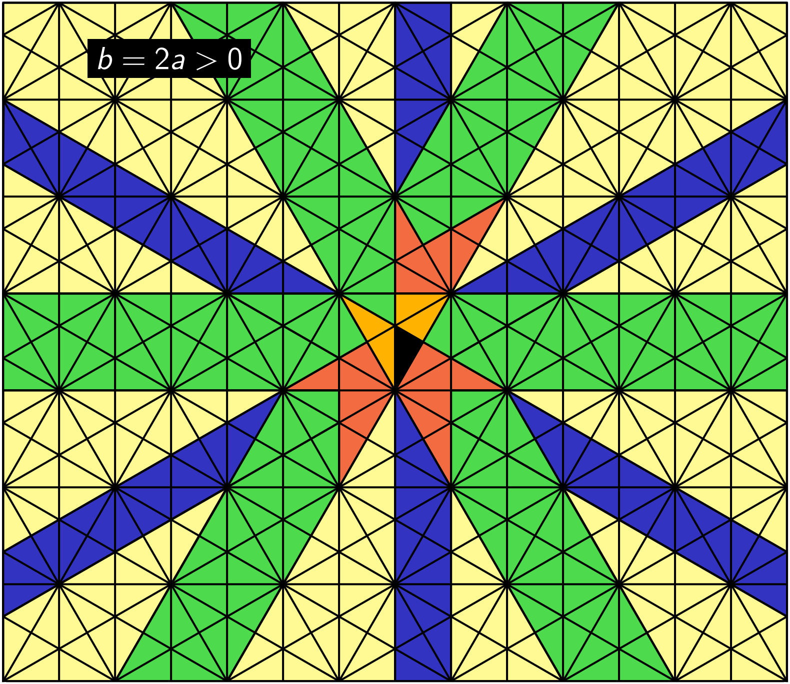
Conjecture (B.)

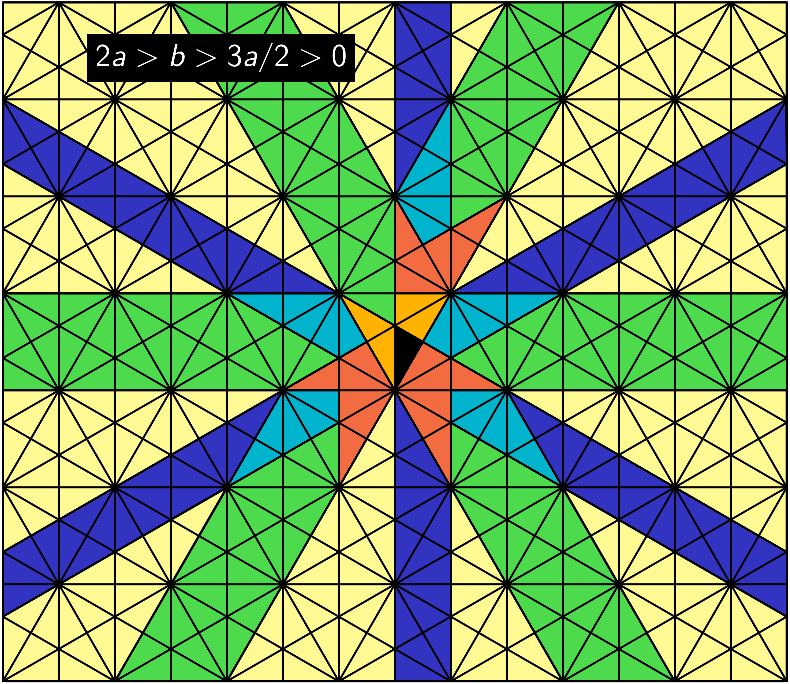
There exists a finite set of (linear) rational hyperplanes $\mathscr{A}$ in V such that:

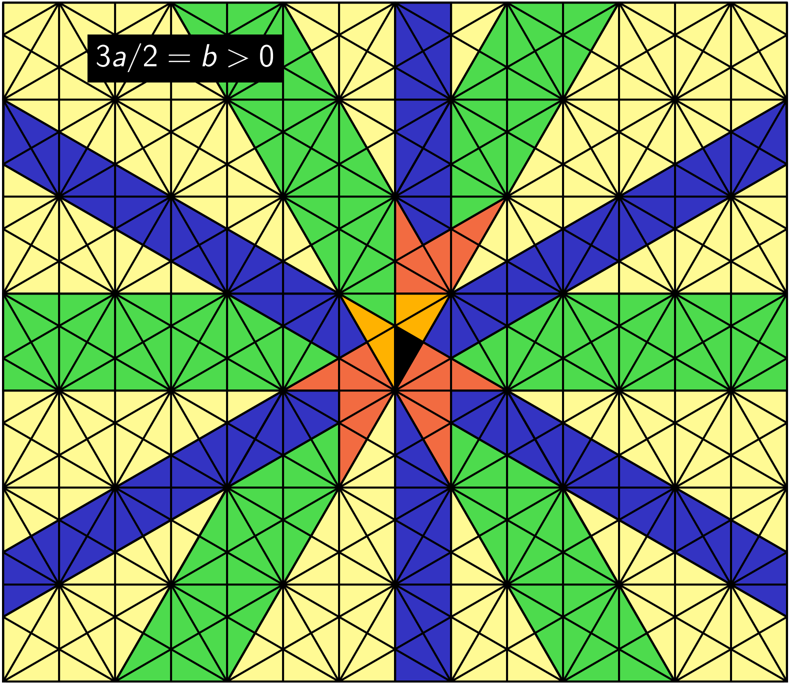
- If φ and φ' belong to the same $\mathscr{A}$-facet in $V^{\#}$, then the left (right, two-sided) cells for (W, S, φ) and (W, S, φ') coincide.
- If $\varphi \in V^{\#}$, then a left (resp. right, two-sided) φ-cell is a **minimal** subset X of W such that:

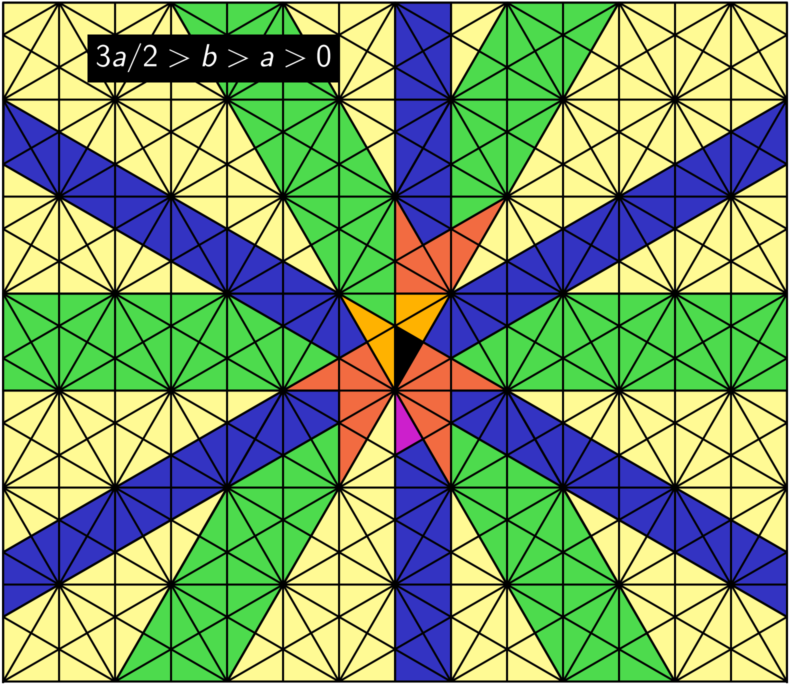
  For each $\mathscr{A}$-chamber $\mathscr{C}$ such that $\varphi \in \overline{\mathscr{C}}$, X is a union of left (resp. right, two-sided) cells for $(W, S, \mathscr{C})$.

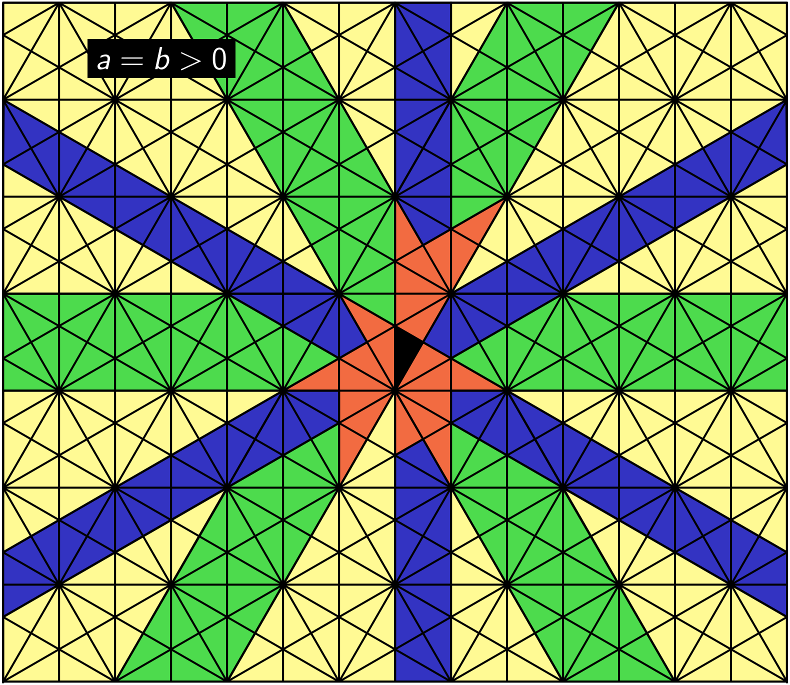

$b > 2a > 0$

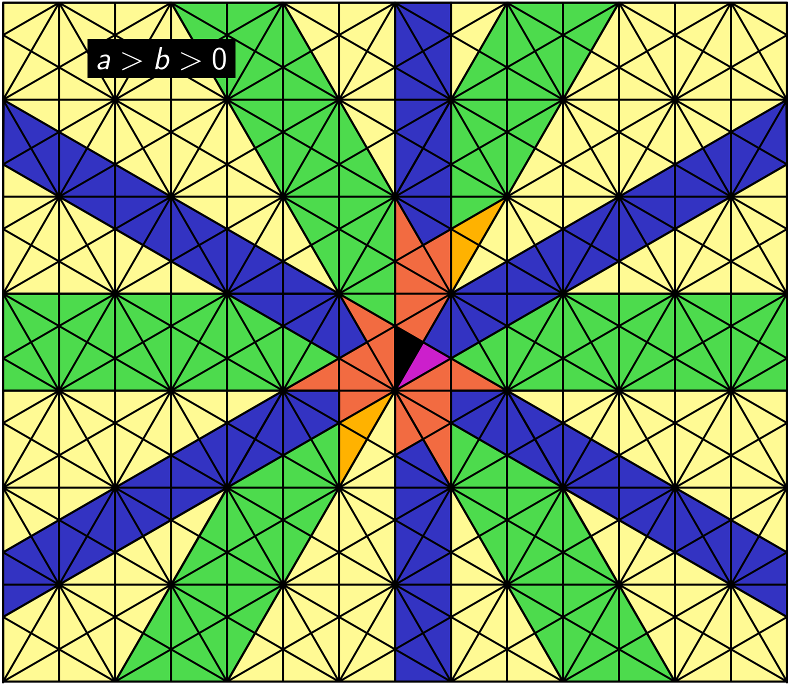

$b = 2a > 0$


$2a > b > 3a/2 > 0$


$3a/2 = b > 0$


$3a/2 > b > a > 0$


$a = b > 0$


$a > b > 0$

Involutions

Involutions

Conjecture (Lusztig).

Every left cell contains at least one "involution".

Involutions

Conjecture (Lusztig).

Every left cell contains at least one "involution".

True for finite Coxeter groups:

Involutions

Conjecture (Lusztig).

Every left cell contains at least one "involution".

True for finite Coxeter groups:

- Equal parameters case $\longrightarrow$ Lusztig

Involutions

Conjecture (Lusztig).

Every left cell contains at least one "involution".

True for finite Coxeter groups:

- Equal parameters case $\longrightarrow$ Lusztig
- Geck (F_4), Iancu-B. 2003 and B. 2008 (type B)

Involutions

Conjecture (Lusztig).

Every left cell contains at least one "involution".

True for finite Coxeter groups:

- Equal parameters case $\longrightarrow$ Lusztig (nice proof: not involving the classification, using the geometry of Schubert varieties).
- Geck (F_4), Iancu-B. 2003 and B. 2008 (type B)

Lusztig's a-function

Lusztig's a-function

From now on, W is finite or affine.

Lusztig's a-function

From now on, W is finite or affine.

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

with $h_{x,y,z} \in R = \mathbb{Z}[\Gamma]$ and $\overline{h_{x,y,z}} = h_{x,y,z}$.

Lusztig's a-function

From now on, W is finite or affine.

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

with $h_{x,y,z} \in R = \mathbb{Z}[\Gamma]$ and $\overline{h_{x,y,z}} = h_{x,y,z}$.

$$\deg : R \longrightarrow \Gamma \cup \{-\infty\},$$
$$\mathrm{val} : R \longrightarrow \Gamma \cup \{+\infty\}.$$

Lusztig's $\mathbf{a}$-function

From now on, W is finite or affine.

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

with $h_{x,y,z} \in R = \mathbb{Z}[\Gamma]$ and $\overline{h_{x,y,z}} = h_{x,y,z}$.

$$\deg : R \longrightarrow \Gamma \cup \{-\infty\},$$
$$\mathrm{val} : R \longrightarrow \Gamma \cup \{+\infty\}.$$

Let

$$\mathbf{a}(z) = \max_{x,y \in W} \deg(h_{x,y,z})$$

Lusztig's $\mathbf{a}$-function

From now on, W is finite or affine.

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z,$$

with $h_{x,y,z} \in R = \mathbb{Z}[\Gamma]$ and $\overline{h_{x,y,z}} = h_{x,y,z}$.

$$\deg : R \longrightarrow \Gamma \cup \{-\infty\},$$

$$\mathrm{val} : R \longrightarrow \Gamma \cup \{+\infty\}.$$

Let

$$\mathbf{a}(z) = \max_{x,y \in W} \deg(h_{x,y,z})$$

Lusztig proved that

$$\mathbf{a}(z) \leqslant \max_{I \subset S, W_I \text{ finite}} \varphi(w_I).$$

- Write
$$h_{x,y,z} \in \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} + R_{<\mathbf{a}(z)}$$
and
$$p^*_{1,z} \in n_z e^{-\Delta(z)} + R_{<-\Delta(z)},$$
with $\Delta(z) \in \Gamma_{\geqslant 0}$, $\gamma_{x,y,z} \in \mathbb{Z}$ and $n_z \in \mathbb{Z}$.

- Write

$$h_{x,y,z} \in \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} + R_{<\mathbf{a}(z)}$$

and

$$p^*_{1,z} \in n_z e^{-\Delta(z)} + R_{<-\Delta(z)},$$

with $\Delta(z) \in \Gamma_{\geqslant 0}$, $\gamma_{x,y,z} \in \mathbb{Z}$ and $n_z \in \mathbb{Z}$.

- $\mathbf{a}(z) \geqslant 0$.

- Write
$$h_{x,y,z} \in \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} + R_{<\mathbf{a}(z)}$$
and
$$p^*_{1,z} \in n_z e^{-\Delta(z)} + R_{<-\Delta(z)},$$
with $\Delta(z) \in \Gamma_{\geqslant 0}$, $\gamma_{x,y,z} \in \mathbb{Z}$ and $n_z \in \mathbb{Z}$.
- $\mathbf{a}(z) \geqslant 0$.
- If $z \in W \setminus \{1\}$, then $\mathbf{a}(z) > 0$.

- Write
$$h_{x,y,z} \in \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} + R_{<\mathbf{a}(z)}$$
and
$$p^*_{1,z} \in n_z e^{-\Delta(z)} + R_{<-\Delta(z)},$$
with $\Delta(z) \in \Gamma_{\geqslant 0}$, $\gamma_{x,y,z} \in \mathbb{Z}$ and $n_z \in \mathbb{Z}$.
- $\mathbf{a}(z) \geqslant 0$.
- If $z \in W \setminus \{1\}$, then $\mathbf{a}(z) > 0$.
- Finally, let
$$\mathscr{D} = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}.$$

Asymptotic algebra

Asymptotic algebra

Let $\mathscr{J} = \bigoplus_{w \in W} \mathbb{Z}\, t_w$ and, if $x, y \in W$, we set

$$t_x \cdot t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

Asymptotic algebra

Let $\mathscr{J} = \bigoplus_{w \in W} \mathbb{Z}\, t_w$ and, if $x, y \in W$, we set

$$t_x \cdot t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

Theorem (Lusztig)

If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then the number of left cells is finite (so $|\mathscr{D}| < \infty$) and $(\mathscr{J}, \cdot)$ is a unitary associative algebra.

Asymptotic algebra

Let $\mathscr{J} = \bigoplus_{w \in W} \mathbb{Z}\, t_w$ and, if $x, y \in W$, we set

$$t_x \cdot t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

Theorem (Lusztig)

If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then the number of left cells is finite (so $|\mathscr{D}| < \infty$) and $(\mathscr{J}, \cdot)$ is a unitary associative algebra. The unit is $\sum_{d \in \mathscr{D}} n_d t_d$. We have

$$t_d t_e = n_d \delta_{d,e} t_d$$

for all $d, e \in \mathscr{D}$.

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d .$$

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures P_1, $P_2, \ldots$, P_{15} hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J} b_C$ and b_C is a central idempotent.*

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures P_1, $P_2, \ldots,$ P_{15} hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$ and b_C is a central idempotent.*
- $\mathscr{J} \simeq \prod_{C \text{ two-sided cell}} \mathscr{J}_C$ *as a ring.*

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures P_1, $P_2, \ldots,$ P_{15} hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$ and b_C is a central idempotent.*
- $\mathscr{J} \simeq \prod_{C \text{ two-sided cell}} \mathscr{J}_C$ *as a ring.*
- *If C is a left cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$.*

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures P_1, $P_2, \ldots,$ P_{15} hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J} b_C$ and b_C is a central idempotent.*
- $\mathscr{J} \simeq \prod_{C \text{ two-sided cell}} \mathscr{J}_C$ *as a ring.*
- *If C is a left cell, then $\mathscr{J}_C = \mathscr{J} b_C$. In particular, it is a left ideal, projective as a $\mathscr{J}$-module.*

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$ and b_C is a central idempotent.*
- $\mathscr{J} \simeq \prod_{C \text{ two-sided cell}} \mathscr{J}_C$ *as a ring.*
- *If C is a left cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$. In particular, it is a left ideal, projective as a $\mathscr{J}$-module.*
- *If C is a left cell, then $\mathscr{J}_{C \cap C^{-1}} = b_C \mathscr{J} b_C$,*

Asymptotic algebra (continued)

If X is a subset of W, we set

$$\mathscr{J}_X = \bigoplus_{w \in X} \mathbb{Z}\, t_w \quad \text{and} \quad b_X = \sum_{d \in \mathscr{D} \cap X} n_d t_d.$$

Theorem (Lusztig)

If Lusztig's conjectures $P_1, P_2, \ldots, P_{15}$ hold, then

- *If C is a two-sided cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$ and b_C is a central idempotent.*
- $\mathscr{J} \simeq \prod_{C \text{ two-sided cell}} \mathscr{J}_C$ *as a ring.*
- *If C is a left cell, then $\mathscr{J}_C = \mathscr{J}\, b_C$. In particular, it is a left ideal, projective as a $\mathscr{J}$-module.*
- *If C is a left cell, then $\mathscr{J}_{C \cap C^{-1}} = b_C\, \mathscr{J}\, b_C$, so $\mathscr{J}_{C \cap C^{-1}}$ is a ring (with unit b_C) isomorphic to $\mathrm{End}_{\mathscr{J}}(\mathscr{J}_C)^{\mathrm{opp}}$.*