Blocks and decomposition numbers for the Brauer algebra

Anton Cox

City University London

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and the framework used to study them was introduced in earlier work

- *Towers of recollement:*
  [CMPX] with Paul Martin, Alison Parker, and Changchang Xi

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The set of allowable diagrams for a given diagram algebra A_n will form a basis for A_n as a vector space. Multiplication of two diagrams is given by concatenation. However, as the result may not lie in the space spanned by allowable diagrams, we may impose additional relations to reduce diagrams to an allowable form.

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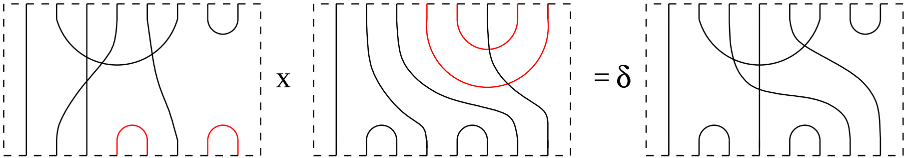

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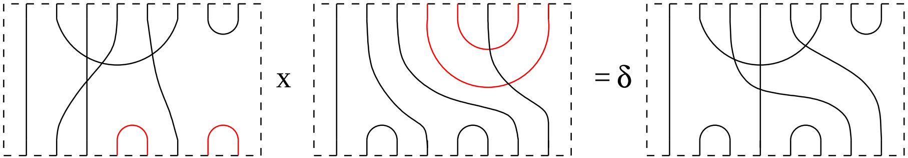

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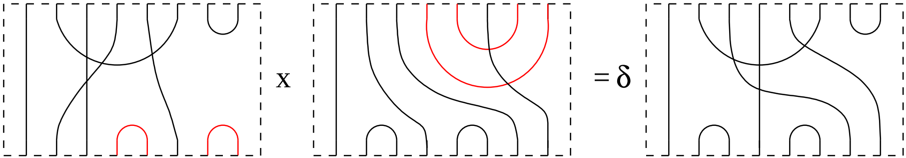

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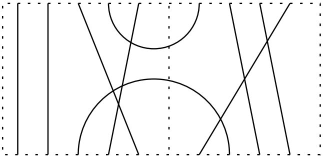

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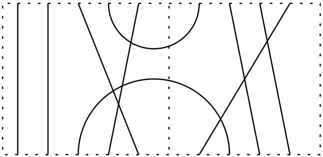

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This contains the group algebra of $\Sigma_r \times \Sigma_s$.

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- As (affine) Hecke algebra quotients. The (affine) Temperley-Lieb and blob algebras can be used to study Hecke algebras.
- In invariant theory. This is the area which will motivate our study of the (walled) Brauer algebras.

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The Brauer algebra $B_r(\delta)$ was introduced to play the role of the symmetric group in a corresponding Schur-Weyl duality for the symplectic and orthogonal groups.

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There is a corresponding Schur-Weyl duality relating the walled Brauer algebra $B_{r,s}(n)$ with the action of $\mathrm{GL}_n(\mathbb{C})$ on 'mixed' tensor space $V^{\otimes r} \otimes (V^*)^{\otimes s}$.

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Physics tells us to expect that answers should (in a suitable sense) be independent of n.

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We also need a suitable combinatorial framework for expressing our results.

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We will give a very brief review of the main features of this.

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In characteristic p the blocks of G are given* by the intersection of the orbits of W_p under the dot action with the set of dominant weights.

Example: SL_3

For SL_3 we have $W = \Sigma_3$ acting on $\mathbb{Z}^2$ with $\rho = (1, 1)$.

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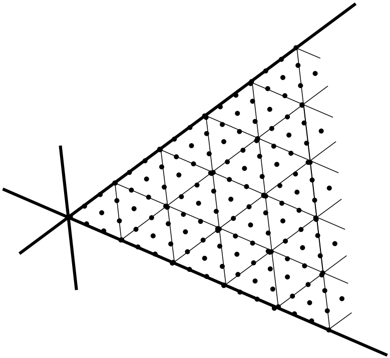

Figure: Dominant alcoves for $SL_3(k)$.

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Henceforth we assume that $\delta \in \mathbb{Z}$, as the Brauer algebras are semisimple in all other cases.

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(Earlier work by Brown, Hanlon and Wales, and Wenzl, showed that the Brauer algebras over $\mathbb{C}$ were semisimple for all non-integral choices of δ. A necessary and sufficient criterion for semisimplicity, over arbitrary k, was given by Rui.)

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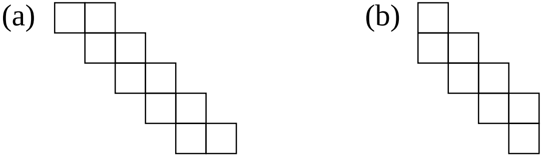

Figure: Two posible configurations of boxes.

Simple modules for $B_n(\delta)$ can be naturally labelled by partitions of n, $n-2, \ldots$

Theorem (CDM 05)

Let $k = \mathbb{C}$ and $\delta \neq 0$. Two simple Brauer modules $L(\lambda)$ and $L(\mu)$ are in the same block if and only if λ and $\lambda \cap \mu$ are balanced and μ and $\lambda \cap \mu$ are balanced.

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This is not a geometric description as we initially wanted. So we will recast the combinatorics.

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Theorem (CDM 09)

Suppose that $k = \mathbb{C}$ and $\delta \in \mathbb{Z}$. If λ and μ lie in the same orbit of W, then the associated decomposition number is given by the corresponding p-KL polynomial evaluated at 1*.*

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Further we can give explicit examples of pairs of weights which show that the reverse implication does not always hold. Thus we have what in Lie theory would be called a linkage principle in positive characteristic.

We can develop a version of all of our theory for $B_n(\delta)$ for $B_{r,s}(\delta)$, with the role of symmetric group combinatorics replaced by $\Sigma_r \times \Sigma_s$ combinatorics and partitions replaced by bipartitions.

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Once again this combinatoric can be reinterpreted geometrically.

Identify a bipartition (λ^L, λ^R) with r and s parts with the element of $\mathbb{Z}^{r+s}$ given by

$$(-\lambda^L_r, -\lambda^L_{r-1}, \ldots, -\lambda^L_1, \lambda^R_1, \lambda^R_2, \ldots, \lambda^R_s)$$

and call the set X^+ of such elements dominant weights.

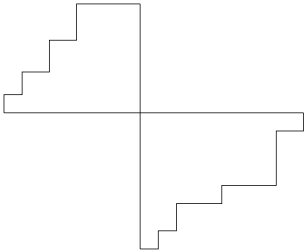

Figure: A dominant weight

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> The study of diagram algebras may lead to more general contexts in which the machinery of algebraic Lie theory can be applied.