Combinatorics of Kazhdan–Lusztig cells

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- $m_{st} = m_{ts} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ for $s \neq t$.

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Examples:

$$W(I_2(m)) = \langle s, t \mid s^2 = t^2 = 1,\ (st)^m = 1 \rangle \qquad \text{dihedral of order } 2m$$

$$W(A_{n-1}) = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = 1,\ \ (s_i s_{i+1})^3 = 1, \\ (s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \end{array} \right\rangle \cong \mathfrak{S}_n.$$

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For finite irreducible Coxeter groups, "unequal parameters" can only occur in types $I_2(m)$ (with m even), B_n and F_4.

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Important aim: Understand $\mathrm{Irr}(\mathcal{H}_k)$ for various $\theta \colon A \to k$.

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IWAHORI (1964) + DIPPER (1990): There is a natural bijection

$$\mathrm{Irr}_k(G, B) \quad \stackrel{1-1}{\longleftrightarrow} \quad \mathrm{Irr}(\mathcal{H}_k)$$

where $\mathcal{H} = \mathcal{H}(W, S, \varphi)$ generic Iwahori–Hecke algebra (suitable φ), $\theta\colon A \to k$ is such that $\theta(\varphi(s)) = q^{c_s} 1_k$ for certain integers $c_s > 0$.

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(More general types of parameters occur when we consider the Harish-Chandra induction of cuspidal representations from Levi subgroups.)

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$$\Gamma_+ := \{\gamma \in \Gamma \mid \gamma > 1\}, \qquad \Gamma = \Gamma_+ \cup \{1\} \cup \Gamma_-$$

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- The map $\bar{\ }: A \to A$, $\bar{\gamma} = \gamma^{-1}$ extends to a ring involution on $\mathcal{H}$

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1}.$$

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- Up to normalisation, the $P^*_{y,w}$ are the famous Kazhdan–Lusztig polynomials. They can be computed using a recursive formula. No general closed formula is known.

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Let $\mathcal{C}$ be a left cell of W. Then the A-modules

$$\mathcal{I}_{\mathcal{C}} = \langle C_y \mid y \leqslant_L w, \text{ for some } w \in \mathcal{C} \rangle_A$$
$$\hat{\mathcal{I}}_{\mathcal{C}} = \langle C_y \mid y \leqslant_L w, \text{ for some } w \in \mathcal{C}, \ y \notin \mathcal{C} \rangle_A$$

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$$[\mathcal{C}]_A := \mathcal{I}_\mathcal{C} / \hat{\mathcal{I}}_\mathcal{C} \quad \text{is a left } \mathcal{H}\text{-module.}$$

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EXAMPLE

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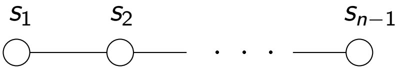

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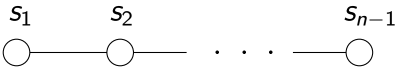

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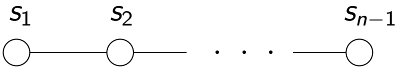

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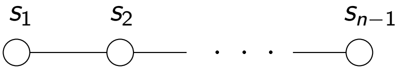

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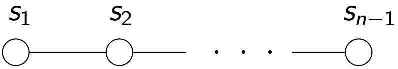

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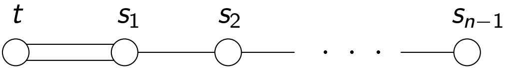
In general, Dipper and James also showed that $\beta_k^\lambda \neq 0$ iff λ is e-regular, i.e., $\mathrm{Irr}(\mathcal{H}_k) = \{D_k^\lambda \mid \lambda \vdash n \text{ is } e\text{-regular}\}$.

EXAMPLE B_n



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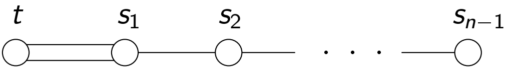

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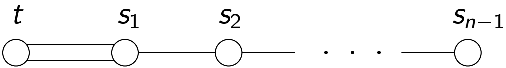

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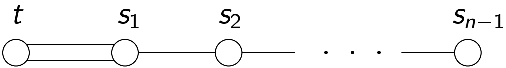

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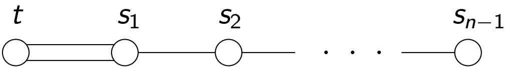

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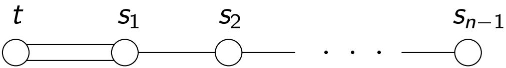
EXAMPLE B_n 

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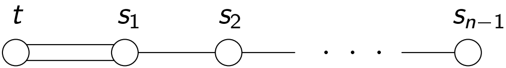

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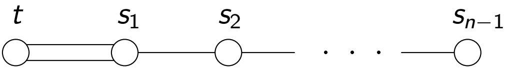

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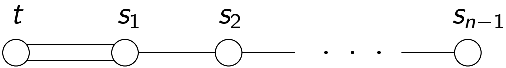

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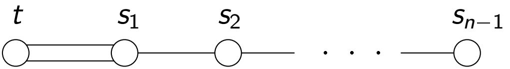

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- Dipper–James–Murphy also formulate a conjecture about when "$\beta^\lambda_k \neq 0$" (much more complicated than "e-regular" in type A): Proved by ARIKI–JACON (2008).

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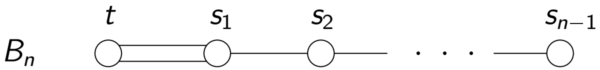

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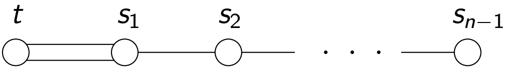

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THE REMAINING CASES

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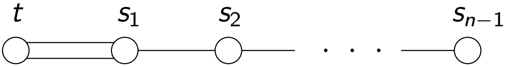

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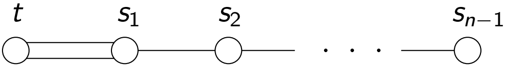

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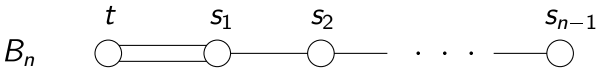

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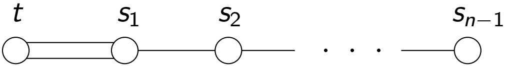

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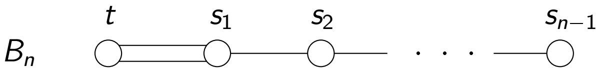

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Evidence for the conjectures.

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- Conjectures 1 and 2 are consistent with analogous results for type $I_2(m)$ (LUSZTIG 2003) and F_4 (GECK 2004).

CONJECTURE 3 (BGIL 2006) Assume that $v^r < V < v^{r+1}$ for some $r \geqslant 0$. Then the Kazhdan-Lusztig basis $\{C_w\}$ is a "cellular basis" in the sense of GRAHAM–LEHRER.

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In particular, this means that there will be a collection of left cells $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ where $\Lambda = \{\text{bipartitions of } n\}$ such that:

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Evidence: 1) B.–G.–I. (2003–2006): Proved if $r \geqslant n-1$ (asymptotic case).
2) Existence of "cellular structure" (in the sense of Graham–Lehrer), for any $\Gamma_+ \subset \Gamma$, can be deduced from Lusztig's conjectures **P1–P15** (GECK 2007).
3) $\Lambda_r^\circ \subseteq \Lambda$ are determined by GECK–JACON (2007) using alternative methods.

$\rightsquigarrow$ different theory of Specht modules for each r.


YOU WANT PROOF?
I'LL GIVE YOU PROOF!
S. harris

Thank You