Littlewood–Richardson polynomials

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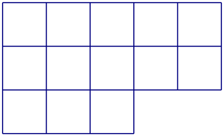

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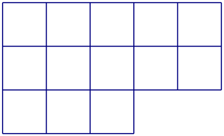

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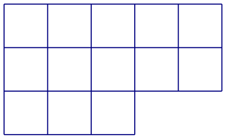

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Then

$$V^{\lambda} \otimes V^{\mu} \cong \bigoplus_{\nu} c_{\lambda\mu}^{\nu} \, V^{\nu}.$$

Here $\quad \ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$.

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In particular,

$$c_{\lambda\mu}^\nu \neq 0 \quad \Longrightarrow \quad |\nu| = |\lambda| + |\mu|.$$

Let n and N be nonnegative integers with $n \leqslant N$ and let $\mathrm{Gr}_{n,N}$ denote the Grassmannian of the n-dimensional vector subspaces of $\mathbb{C}^N$. The cohomology ring $H^*(\mathrm{Gr}_{n,N})$ has a basis of the Schubert classes σ_λ parameterized by all diagrams λ contained in the $n \times m$ rectangle, $m = N - n$.

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We have

$$\sigma_\lambda \, \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \, \sigma_\nu .$$

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Given any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define the corresponding

monomial symmetric function by

$$m_\lambda(x) = \sum_\sigma x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(n)}^{\lambda_n},$$

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The algebra of symmetric functions Λ is defined as the $\mathbb{Q}$-span of all monomial symmetric functions.

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$$h_k(x) = \sum_{|\lambda| = k} m_\lambda(x) = \sum_{i_1 \geqslant \cdots \geqslant i_k \geqslant 1} x_{i_1} \ldots x_{i_k}.$$

Schur functions

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Given a diagram λ, a reverse λ-tableau T is obtained by filling in the boxes of λ with the numbers $1, 2, \ldots$ in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of λ we let $T(\alpha) = T(i, j)$ denote the entry of T in the box α.

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Hence

$$s_{(2,1)}(x) = \sum_{i \geqslant j,\ i > k} x_i\, x_j\, x_k.$$

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$$s_{(k)}(x) = \sum_{i_1 \geqslant \cdots \geqslant i_k \geqslant 1} x_{i_1} \dots x_{i_k},$$

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The relation

$$s_\lambda(x)\, s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu\, s_\nu(x)$$

defines the Littlewood–Richardson coefficients $c_{\lambda\mu}^\nu$.

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Now:

A couple of dozens of versions of the LR rule, $c^{\nu}_{\lambda\mu}$ counts

tableaux, trees, hives, honeycombs, cartons, puzzles, ...

Knutson–Tao–Woodward puzzles

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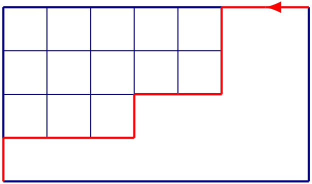

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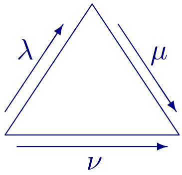

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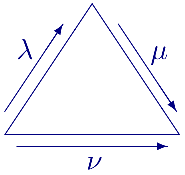
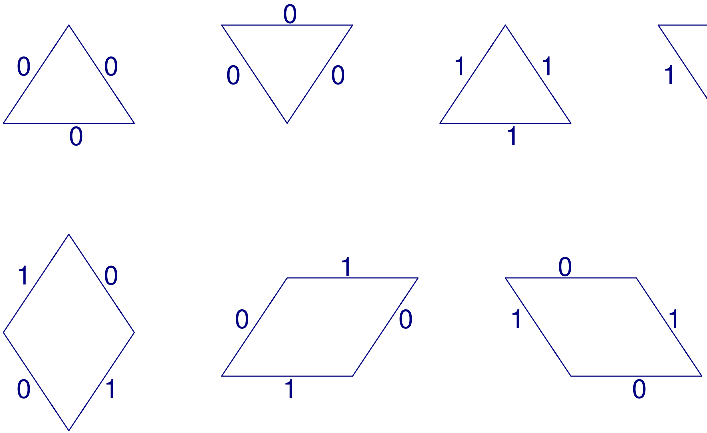


Theorem [KTW '03]. The Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ equals the number of triangular puzzles which can be obtained with the use of the following set of unit puzzle pieces.

Puzzle pieces



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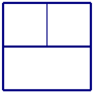

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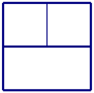
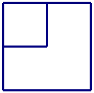


$\lambda \longrightarrow$ 1 0 0 1



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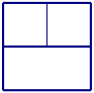

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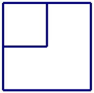

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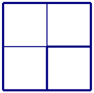

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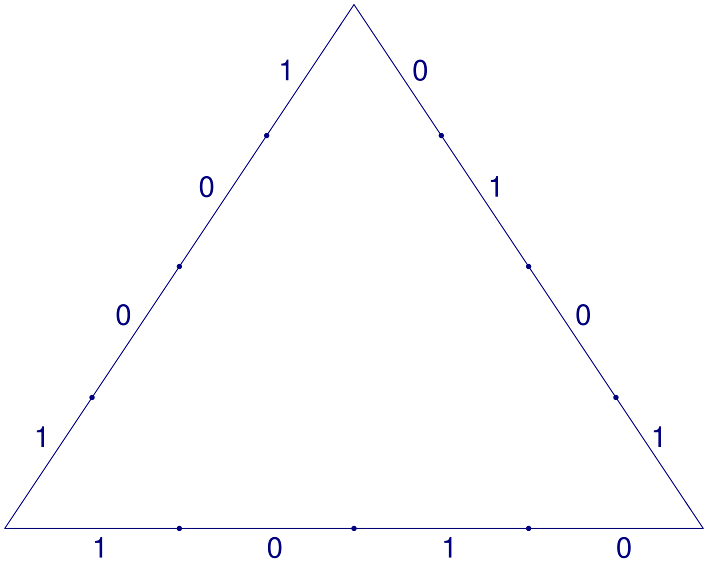
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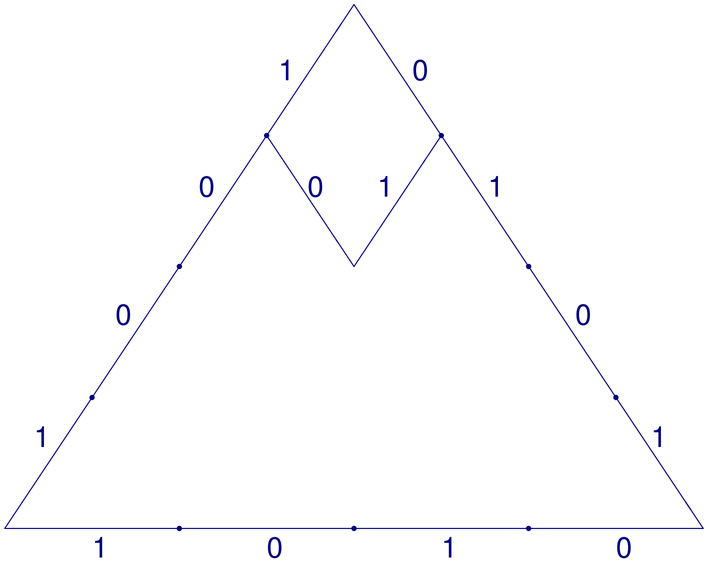
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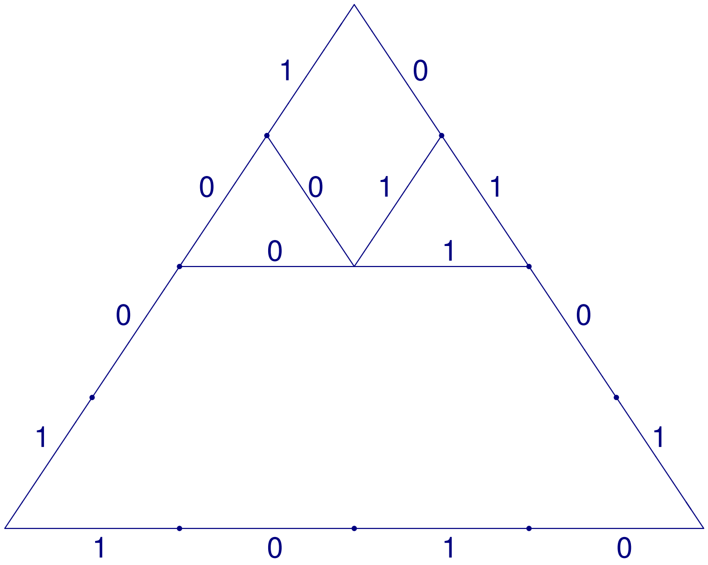
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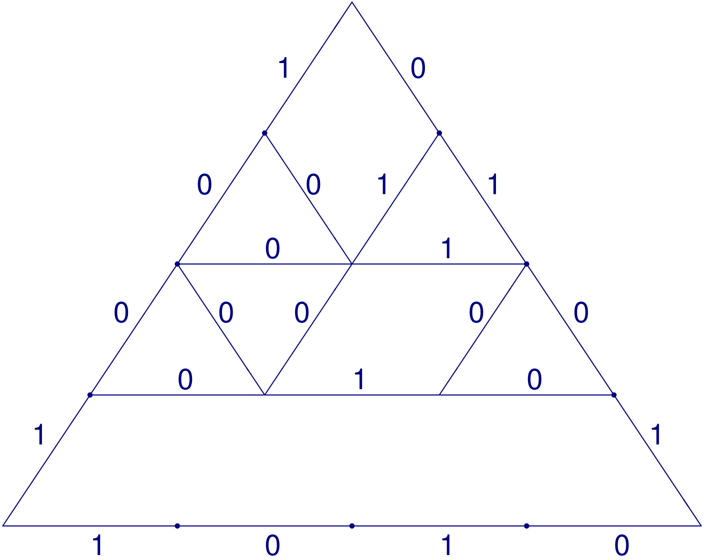
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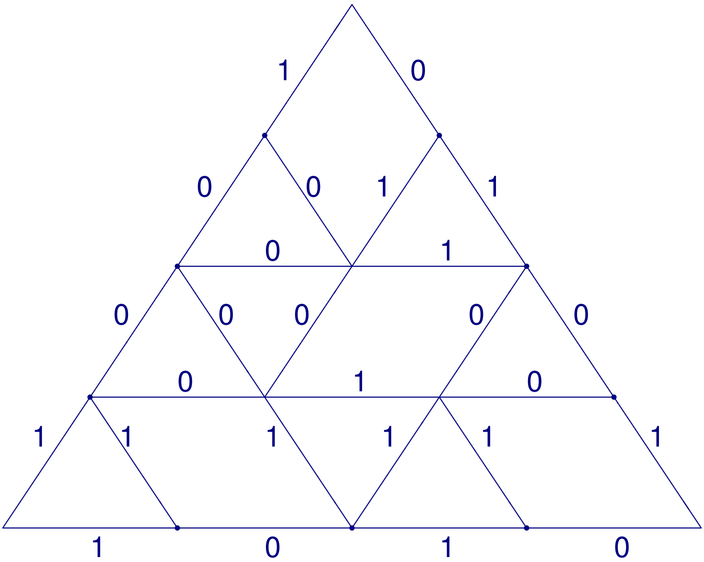
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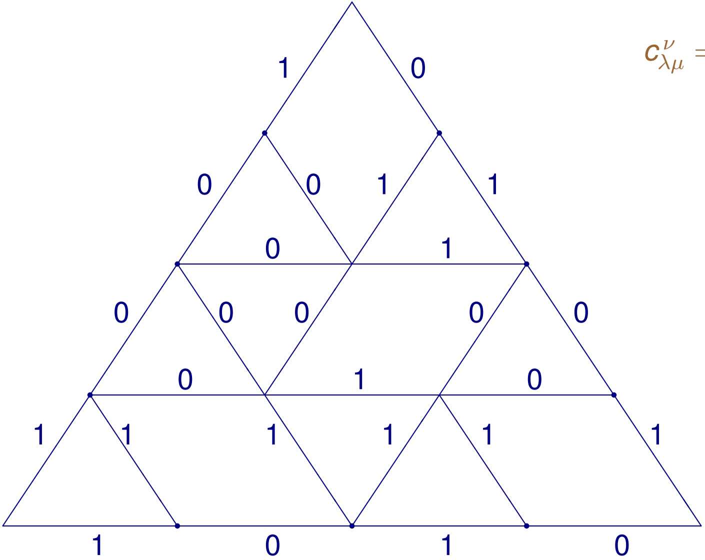

$c^{\nu}_{\lambda\mu} = 1.$

A tableau version of the LR rule

Let R denote a sequence of diagrams

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

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$\rho \to \sigma$ means σ is obtained from ρ by adding one box.

Let r_i denote the row number of the box added to $\rho^{(i-1)}$.

The sequence $r_1\, r_2\, \ldots\, r_l$ is the Yamanouchi symbol of R.

Example. Let

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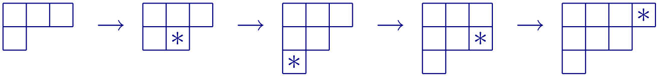

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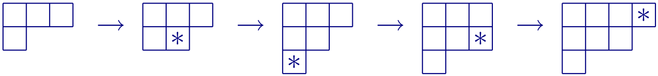

the Yamanouchi symbol is $2\,3\,2\,1$.

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A reverse λ-tableau T is called ν-bounded if the entries in the top row do not exceed the respective column lengths of ν:

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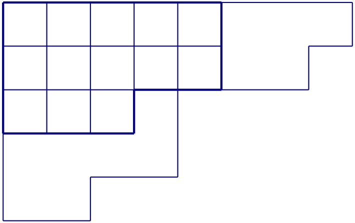

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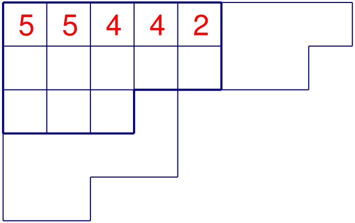
Maximal entries:



Theorem. The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ equals the number of common elements in the two sets:

$\Big\{$column words of the ν-bounded reverse λ-tableaux$\Big\}$ and

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Theorem. The Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ equals the number of common elements in the two sets:

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Remarks.

- This is a particular case of a more general theorem.
- The theorem is equivalent to the puzzle rule (T. Tao).

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$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

The set of column words is

$$\Big\{ 2\,3\,2, \quad 1\,3\,2, \quad 2\,3\,1, \quad 1\,3\,1, \quad 1\,2\,2, \quad 1\,2\,1 \Big\}.$$

The sequences from $(2, 1)$ to $(3, 2, 1)$:

$(2, 1) \to (3, 1) \to (3, 2) \to (3, 2, 1)$

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The set of the Yamanouchi symbols is

$$\Big\{ 1\,2\,3, \quad 1\,3\,2, \quad 2\,1\,3, \quad 2\,3\,1, \quad 3\,1\,2, \quad 3\,2\,1 \Big\}.$$

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Hence $c_{\lambda\mu}^{\nu} = 2$.

Double symmetric functions

The elements of the algebra of symmetric functions Λ can be viewed as sequences of symmetric polynomials:

$$\sum_{i=1}^{\infty} x_i^k \qquad \longrightarrow$$

$$x_1^k, \quad x_1^k + x_2^k, \quad \dots, \quad x_1^k + x_2^k + \cdots + x_n^k, \quad \dots$$

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The polynomials in such a sequence are compatible with the evaluation homomorphisms

$$\varphi_n : P(x_1, \dots, x_n) \mapsto P(x_1, \dots, x_{n-1}, 0).$$

Let $a = (a_i)$, $i \in \mathbb{Z}$, be a sequence of variables.

Denote by Λ_n the ring of symmetric polynomials in $x_1, \dots, x_n$

with coefficients in $\mathbb{Q}[a]$.

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$$\varphi_n : \Lambda_n \to \Lambda_{n-1}, \qquad P(x_1, \dots, x_n) \mapsto P(x_1, \dots, x_{n-1}, a_n).$$

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The ring Λ^a of double symmetric functions is formed by such

sequences of polynomials. The sequences can also be

regarded as formal series.

Examples. We have

$$\varphi_n : \sum_{i=1}^{n}(x_i^k - a_i^k) \mapsto \sum_{i=1}^{n-1}(x_i^k - a_i^k)$$

hence

$$p_k(x \,\|\, a) = \sum_{i=1}^{\infty}(x_i^k - a_i^k) \in \Lambda^a,$$

the double power sums symmetric function.

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Λ^a is the ring of polynomials in

$p_1(x \,\|\, a), \quad p_2(x \,\|\, a), \quad \dots .$

with coefficients in $\mathbb{Q}[a]$. Note that $\Lambda^0 = \Lambda$.

Double Schur functions

For any diagram λ define the double Schur function by

$$s_\lambda(x\,\|\,a) = \sum_T \prod_{\alpha\in\lambda} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}),$$

summed over the reverse λ-tableaux T,

$c(\alpha) = j - i$ is the content of the box $\alpha = (i, j)$.

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The double Schur functions form a basis of Λ^a over $\mathbb{Q}[a]$.

Example. For $\lambda = (2, 1)$ the reverse tableaux are

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{with} \quad i \geqslant j \quad \text{and} \quad i > k$$

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Hence

$$s_{(2,1)}(x \,\|\, a) = \sum_{i \geqslant j,\ i > k} (x_i - a_i)(x_j - a_{j-1})(x_k - a_{k+1}).$$

Set $h_k(x \,\|\, a) = s_{(k)}(x \,\|\, a), \qquad e_k(x \,\|\, a) = s_{(1^k)}(x \,\|\, a).$

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Tableaux

i_1	i_2	$\cdots$	i_k

i_1
i_2
$\vdots$
i_k

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Tableaux

$$\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \quad\cdots\quad & i_k \\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array}$$

Double complete and elementary symmetric functions:

$$h_k(x \,\|\, a) = \sum_{i_1 \geqslant \cdots \geqslant i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k - k + 1}),$$

$$e_k(x \,\|\, a) = \sum_{i_1 > \cdots > i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k + k - 1}).$$

Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a) \in \mathbb{Q}[a]$ by

$$s_\lambda(x \,\|\, a)\, s_\mu(x \,\|\, a) = \sum_{\nu} c_{\lambda\mu}^{\nu}(a)\, s_\nu(x \,\|\, a).$$

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Properties.

- $c_{\lambda\mu}^{\nu}(a) \neq 0$ only if $|\nu| \leqslant |\lambda| + |\mu|$.

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- $c_{\lambda\mu}^{\nu}(a) = c_{\mu\lambda}^{\nu}(a)$.
- $c_{\lambda\mu}^{\nu}(a) \neq 0$ only if $\lambda \subseteq \nu$ and $\mu \subseteq \nu$.

Calculation of $c_{\lambda\mu}^{\nu}(a)$

Given a sequence R from μ to ν with the Yamanouchi symbol $r_1\, r_2 \dots r_l$, introduce the set $\mathcal{T}(\lambda, R)$ of barred reverse λ-tableaux T with entries from $\{1, 2, \dots\}$ such that T contains entries $r_1,\ r_2, \dots, r_l$ listed in the column order.

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We will distinguish these entries by barring each of them.

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We will distinguish these entries by barring each of them.

An element $T \in \mathcal{T}(\lambda, R)$ is a pair consisting of a reverse λ-tableau and a sequence of barred entries compatible with R.

Example. Let R be the sequence

$$(3,1) \to (3,2) \to (3,2,1) \to (3,3,1) \to (4,3,1)$$

so that the Yamanouchi symbol is $2\,3\,2\,1$.

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Let $\lambda = (5,5,3)$. The barred λ-tableau

7	7	4	$\bar{2}$	2
4	$\bar{3}$	2	1	$\bar{1}$
$\bar{2}$	1	1		

belongs to $\mathcal{T}(\lambda, R)$.

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Given a sequence of diagrams

$$R: \qquad \mu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

set $\rho(\alpha) = \rho^{(i)}$ for any box α occupied by an unbarred entry of T, between $\bar{r}_i$ and $\bar{r}_{i+1}$ in column order.

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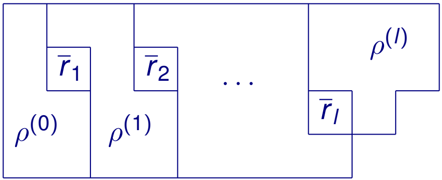
set $\rho(\alpha) = \rho^{(i)}$ for any box α occupied by an unbarred entry of T, between $\bar{r}_i$ and $\bar{r}_{i+1}$ in column order.

The barred entries $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_l$ of T divide the tableau into regions marked by the elements of the sequence R:



Theorem (Kreiman & M. ’07, independently). We have

$$c^{\nu}_{\lambda\mu}(a) = \sum_{R} \sum_{T} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\,\text{unbarred}}} \left(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)} \right),$$

summed over all sequences R from μ to ν and all ν-bounded reverse λ-tableaux $T \in \mathcal{T}(\lambda, R)$.

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summed over all sequences R from μ to ν and all ν-bounded reverse λ-tableaux $T \in \mathcal{T}(\lambda, R)$. Moreover, in each factor $\rho(\alpha)_{T(\alpha)} > c(\alpha)$.

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Remarks.

- If $|\nu| = |\lambda| + |\mu|$ then this is a version of the LR rule.

Theorem (Kreiman & M. '07, independently). We have

$$c^{\nu}_{\lambda\mu}(a) = \sum_{R} \sum_{T} \prod_{\substack{\alpha \in \lambda \\ T(\alpha)\ \text{unbarred}}} \left(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)} \right),$$

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Remarks.

- If $|\nu| = |\lambda| + |\mu|$ then this is a version of the LR rule.
- $c^{\nu}_{\lambda\mu}(a)$ is Graham-positive: it is a polynomial in the differences $a_i - a_j$, $i < j$, with positive integer coefficients.

Example. Calculation of $c_{\lambda\mu}^{\nu}(a)$,

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3	1
2	

3	1
1	

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1	

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3	1
2	

3	1
1	

2	1
1	

There are two sequences

$R_1 : \qquad (3, 1) \to (4, 1) \to (4, 1, 1)$ and

$R_2 : \qquad (3, 1) \to (3, 1, 1) \to (4, 1, 1)$

with the respective Yamanouchi symbols $1\,3$ and $3\,1$.

$\mathcal{T}(\lambda, R_1)$ contains one barred tableau

$\overline{3}$	1
$\overline{1}$	

with $\quad T(\alpha) = 1, \quad \rho(\alpha) = (4, 1, 1), \quad c(\alpha) = 1,$

contributing $\quad a_{T(\alpha) - \rho(\alpha)_{T(\alpha)}} - a_{T(\alpha) - c(\alpha)} = a_{-3} - a_0.$

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$\mathcal{T}(\lambda, R_2)$ contains two barred tableaux with contributions

$$\begin{array}{|c|c|}\hline \overline{3} & \overline{1} \\ \hline 1 \\ \cline{1-1}\end{array} \qquad a_{-2} - a_2\,, \qquad\qquad \begin{array}{|c|c|}\hline \overline{3} & \overline{1} \\ \hline 2 \\ \cline{1-1}\end{array} \qquad a_1 - a_3\,.$$

$\mathcal{T}(\lambda, R_1)$ contains one barred tableau

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Hence $\quad c_{\lambda\mu}^{\nu}(a) = a_{-3} - a_0 + a_{-2} - a_2 + a_1 - a_3.$

Example. For the product of the double Schur functions $s_{(2)}(x\,\|\,a)$ and $s_{(2,1)}(x\,\|\,a)$ we have

$$
\begin{aligned}
& s_{(2)}(x\,\|\,a)\, s_{(2,1)}(x\,\|\,a) \\
& \quad = s_{(4,1)}(x\,\|\,a) + s_{(3,2)}(x\,\|\,a) + s_{(3,1,1)}(x\,\|\,a) + s_{(2,2,1)}(x\,\|\,a) \\
& \quad + (a_{-1} - a_0)\, s_{(2,1,1)}(x\,\|\,a) + (a_{-1} - a_2)\, s_{(2,2)}(x\,\|\,a) \\
& \quad + (a_{-1} - a_2 + a_{-2} - a_0)\, s_{(3,1)}(x\,\|\,a) \\
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Proof of the theorem. Calculate $c^{\nu}_{\lambda\mu}(a)$ by induction on $|\nu| - |\mu|$.

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Proof of the theorem. Calculate $c^{\nu}_{\lambda\mu}(a)$ by induction on $|\nu| - |\mu|$.

Starting point: the Vanishing Theorem (A. Okounkov, ’96):

$$s_\lambda(a_\rho \,\|\, a) = 0 \quad \text{unless} \quad \lambda \subseteq \rho, \qquad\qquad s_\lambda(a_\lambda \,\|\, a) \neq 0,$$

where

$$a_\rho = (a_{1-\rho_1},\, a_{2-\rho_2},\, \dots).$$

Hence, if $R = \{\mu\}$ is a one-term sequence, then

$$c^{\mu}_{\lambda\mu}(a) = s_{\lambda}(a_{\mu} \,\|\, a), \qquad a_{\mu} = (a_{1-\mu_1}, a_{2-\mu_2}, \dots),$$

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Then use the recurrence

$$c_{\lambda\mu}^{\nu}(a) = \frac{1}{|a_{\nu}| - |a_{\mu}|} \left(\sum_{\mu \to \mu^{+}} c_{\lambda\mu^{+}}^{\nu}(a) - \sum_{\nu^{-} \to \nu} c_{\lambda\mu}^{\nu^{-}}(a) \right),$$

where $|a_{\nu}| - |a_{\mu}| = \sum_{i \geqslant 1} \Big((a_{\nu})_i - (a_{\mu})_i \Big)$ (M. & Sagan, ’99).

Knutson–Tao puzzles

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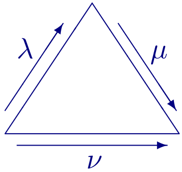

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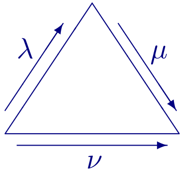
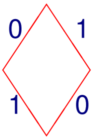
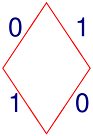


Theorem [KT '03]. The Littlewood–Richardson polynomial $c^{\nu}_{\lambda\mu}(a)$ equals the sum of weights of triangular puzzles, where an additional puzzle piece can be used.

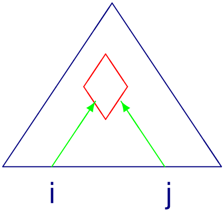
Additional puzzle piece



Additional puzzle piece



Each occurrence of this puzzle piece contributes a factor by the rule:



$\longrightarrow \quad a_{i-m} - a_{j-m}$

Quantum immanants (Okounkov, ’96)

Let $\ell(\lambda) \leqslant n$ and $k = |\lambda|$. Consider the irreducible representation V_λ of the symmetric group $\mathfrak{S}_k$.

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$$\phi_\mathcal{U} = \sum_{s \in \mathfrak{S}_k} (s \cdot v_\mathcal{U}, v_\mathcal{U}) \cdot s^{-1} \in \mathbb{C}[\mathfrak{S}_k].$$

Consider the natural action of the symmetric group $\mathfrak{S}_k$ on the tensor product of k copies of $\mathbb{C}^n$. Denote by $\Phi_\mathcal{U}$ the image of $\phi_\mathcal{U}$ in $(\mathrm{Mat}_n)^{\otimes k}$.

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where $a, b \in \{1, \dots, n\}$.

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For any $c \in \mathbb{C}$ set

$$E - c = \sum_{a,b=1}^{n} e_{ab} \otimes \left(E_{ab} - \delta_{ab}\, c\right) \in \mathrm{Mat}_n \otimes \mathrm{U}(\mathfrak{gl}_n).$$

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The tensor products of the form $(E - c_1) \otimes \ldots \otimes (E - c_k)$ are

regarded as elements of the algebra

$$\mathrm{Mat}_n \otimes \ldots \otimes \mathrm{Mat}_n \otimes \mathrm{U}(\mathfrak{gl}_n).$$

The quantum immanant $\mathbb{S}_\lambda$ associated with λ is the element of $\mathrm{U}(\mathfrak{gl}_n)$ defined by

$$\mathbb{S}_\lambda = \frac{1}{H_\lambda} \operatorname{tr} (E - c_1) \otimes \ldots \otimes (E - c_k) \cdot \Phi_{\mathcal{U}},$$

where the trace is taken over all matrix factors,

- $\mathcal{U}$ is a standard λ-tableau,
- $c_r = j - i$ if r occupies the box (i, j) in $\mathcal{U}$,
- $H_\lambda = k!/\dim V_\lambda$ is the product of the hooks of λ.

Examples. Quantum minors (Capelli elements)

$$\mathbb{S}_{(1^k)} = \sum_{a_1 < \dots < a_k} \sum_{p \in \mathfrak{S}_k} \operatorname{sgn} p \cdot E_{a_1, a_{p(1)}} \dots (E + k - 1)_{a_k, a_{p(k)}}.$$

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Quantum permanents

$$\mathbb{S}_{(k)} = \sum_{a_1 \leqslant \dots \leqslant a_k} \frac{1}{\alpha_1! \dots \alpha_n!} \sum_{p \in \mathfrak{S}_k} E_{a_1, a_{p(1)}} \dots (E - k + 1)_{a_k, a_{p(k)}},$$

where α_i is the multiplicity of i in $a_1, \dots, a_k$, each $a_r \in \{1, \dots, n\}$.

The quantum immanants $\mathbb{S}_\lambda$ with $\ell(\lambda) \leqslant n$ form a basis of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$.

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Corollary. $f_{\lambda\mu}^{\nu} = c_{\lambda\mu}^{\nu}(a)$ for the specialization $a_i = -i$ for $i \in \mathbb{Z}$.

The coefficient $f_{\lambda\mu}^{\nu}$ is zero unless $\lambda, \mu \subseteq \nu$. If $\lambda, \mu \subseteq \nu$ then

$$f_{\lambda\mu}^{\nu} = \sum_{R} \sum_{T} \prod_{\substack{\alpha \in \lambda \\ T(\alpha) \text{ unbarred}}} \Big(\rho(\alpha)_{T(\alpha)} - c(\alpha) \Big),$$

summed over all sequences R from μ to ν and all ν-bounded reverse λ-tableaux $T \in \mathcal{T}(\lambda, R)$. In particular, the $f_{\lambda\mu}^{\nu}$ are nonnegative integers.

Example. For any $n \geqslant 3$ we have

$$\mathbb{S}_{(2)}\,\mathbb{S}_{(2,1)} = \mathbb{S}_{(4,1)} + \mathbb{S}_{(3,2)} + \mathbb{S}_{(3,1,1)} + \mathbb{S}_{(2,2,1)}$$

$$+ \mathbb{S}_{(2,1,1)} + 5\,\mathbb{S}_{(3,1)} + 3\,\mathbb{S}_{(2,2)} + 3\,\mathbb{S}_{(2,1)}.$$

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Equivariant Schubert calculus on the Grassmannian

The torus $T = (\mathbb{C}^*)^N$ acts naturally on $\mathrm{Gr}_{n,N}$. The equivariant cohomology ring $H^*_T(\mathrm{Gr}_{n,N})$ is a module over $\mathbb{Z}[t_1, \ldots, t_N] = H^*_T(\{pt\})$.

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It has a basis of the equivariant Schubert classes σ_λ parameterized by all diagrams λ contained in the $n \times m$ rectangle, $m = N - n$.

Corollary. We have

$$\sigma_\lambda \, \sigma_\mu = \sum_\nu d_{\lambda\mu}^\nu \, \sigma_\nu,$$

where $d_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu(a)$ with the sequence a specialized as follows:

$$a_{-m+1} = -t_1, \quad \dots, \quad a_n = -t_N,$$

and $a_i = 0$ for all remaining values of i.

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The $d_{\lambda\mu}^\nu$ are polynomials in the $t_i - t_j$, $i > j$ with positive integer coefficients (the positivity property, Graham '01).

The coefficients $d^{\nu}_{\lambda\mu}$, regarded as polynomials in the a_i, are independent of n and m, as soon as the inequalities $n \geqslant \lambda'_1 + \mu'_1$ and $m \geqslant \lambda_1 + \mu_1$ hold (the stability property).

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Remark. The puzzle rule of Knutson and Tao gives a manifestly positive formula for the $d^{\nu}_{\lambda\mu}$, while the tableau rule is manifestly stable.

Example. For any $n \geqslant 3$ and $m \geqslant 4$ we have

$$
\begin{aligned}
\sigma_{(2)}\,\sigma_{(2,1)} &= \sigma_{(4,1)} + \sigma_{(3,2)} + \sigma_{(3,1,1)} + \sigma_{(2,2,1)} \\
&+ (t_m - t_{m-1})\,\sigma_{(2,1,1)} + (t_{m+2} - t_{m-1})\,\sigma_{(2,2)} \\
&+ (t_{m+2} - t_{m-1} + t_m - t_{m-2})\,\sigma_{(3,1)} \\
&+ (t_{m+2} - t_{m-1})\,(t_m - t_{m-1})\,\sigma_{(2,1)}.
\end{aligned}
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Dimensions of skew diagrams

Let $\mu \subseteq \lambda$ be two diagrams. The skew diagram $\theta = \lambda/\mu$ is the set-theoretical difference of the diagrams λ and μ:

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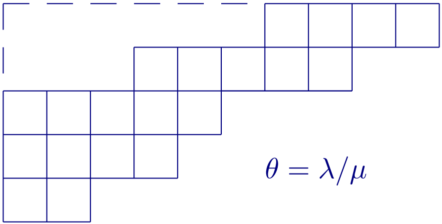
Example. $\lambda = (10, 8, 5, 4, 2)$ and $\mu = (6, 3)$:



If θ has $n = |\theta|$ boxes, then a standard θ-tableau is obtained by filling the boxes bijectively with the numbers $\{1, 2, \ldots, n\}$ in such a way that the entries increase along the rows and down the columns.

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Set

$$H_\theta = \frac{|\theta|!}{\dim \theta}.$$

If θ is normal (nonskew), then H_θ coincides with the product of the hooks of θ due to the hook formula.

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4	2	1	
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Hence $H_\theta = 6 \cdot 4^2 \cdot 3 \cdot 2 \cdot 1^3 = 576$ and $\dim \theta = 70$.

If $\theta = \theta_1 \sqcup \cdots \sqcup \theta_r$, then $H_\theta = H_{\theta_1} \dots H_{\theta_r}$.

Example. Let $\theta = (3,2)/(1)$. The standard θ-tableaux are

	1	2
3	4	

	1	3
2	4	

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2	3	

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Hence $\quad \dim\theta = 5 \quad$ and $\quad H_\theta = 24/5$.

Corollary. We have

$$c^{\nu}_{\lambda\mu} = \sum_{\rho} (-1)^{|\nu/\rho|} \frac{H_\rho}{H_{\nu/\rho}\, H_{\rho/\lambda}\, H_{\rho/\mu}},$$

summed over the diagrams ρ which contain both λ and μ, and are contained in ν.

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Hence

$$c^{(3,2,1)}_{(2,1)(2,1)} = -3 + 8 + 12 + 8 - 24 - 20 - 24 + 45 = 2.$$