Basic set for the alternating group

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- For $\chi \in \mathrm{Irr}(G)$, define $\chi^{p\text{-reg}}(g) = \begin{cases} \chi(g) & \text{if g is p-reg} \\ 0 & \text{otherwise} \end{cases}$.
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  (a p-regular element is an element of G of order prime to p).
- For $S \subseteq \mathrm{Irr}(G)$, put

$$S^{p\text{-reg}} = \{\chi^{p\text{-reg}} \mid \chi \in S\}.$$

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Main problem in p-modular representation theory

One of the main questions in p-modular representation theory of finite groups is to compute the p-decomposition matrix.

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Definition

A subset $B \subseteq \mathrm{Irr}(G)$ is called a p-Basic set of G if $B^{p\text{-reg}}$ is a $\mathbb{Z}$-basis of the ring $\mathbb{Z}\,\mathrm{IBr}_p(G)$.

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- If $|P|$ divides $\chi(1)$, then
  - $\chi^{p\text{-reg}} \in \operatorname{IBr}_p(G)$.
  - If B is a p-basic, then $\chi \in B$.

An example

The character table of $G = \mathfrak{A}_5$ is

	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(1,3,2,4,5)$
χ_1	1	1	1	1	1
χ_2	3	-1	0	z	z^{-1}
χ_3	3	-1	0	z^{-1}	z
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

with $z = \frac{1}{2}(1 + \sqrt{5})$.

An example

$p = 2$

	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(1,3,2,4,5)$
$\chi_1^{\text{2-reg}}$	1	1	1	1	1
$\chi_2^{\text{2-reg}}$	3	-1	0	z	z^{-1}
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- Then $B = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ is a 2-basic set of $\mathfrak{A}_5$.

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with $\psi \in \mathbb{N}\,\mathrm{IBr}_2(G)$.
- Put $\sigma : \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(\sqrt{5}), a + b\sqrt{5} \mapsto a - b\sqrt{5}$.

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$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Basic sets problem

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Basic set problem

Does any finite group have a p-basic set for all prime p?

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- The sporadic groups have a p-basic set for every p (GAP).

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Question

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We only consider here the case that p is odd.

Characters and classes of $\mathfrak{S}_n$

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- Moreover, we have:

$$\chi_\lambda \otimes \varepsilon = \chi_{\lambda^*},$$

where λ^* denotes the conjugate partition of λ.

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  - Put $\overline{\lambda} = (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 1)$.
  - The class $[\overline{\lambda}]$ splits into two classes $[\overline{\lambda}_\pm]$ of $\mathfrak{A}_n$.
  - $\rho_{\lambda,\pm}$ differ only on $[\overline{\lambda}_\pm]$. There is s_λ and t_λ such that

$$\rho_{\lambda,\pm}([\overline{\lambda}_+]) = s_\lambda \pm t_\lambda \quad \text{and} \quad \rho_{\lambda,\pm}([\overline{\lambda}_-]) = s_\lambda \mp t_\lambda.$$

Restriction of p-basic set

Idea

Let $B \subseteq \mathrm{Irr}(\mathfrak{S}_n)$ be a p-basic set. Define

$$B_{\mathfrak{A}_n} = \{\chi \in \mathrm{Irr}(\mathfrak{A}_n) \mid \langle \mathrm{Res}(\psi), \chi \rangle_{\mathfrak{A}_n} \neq 0 \text{ for } \psi \in B\}.$$

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- The restriction $B_{\Gamma,\mathfrak{A}_n}$ is not a p-basic set of $\mathfrak{A}_n$!!

Restriction of p-basic set

Theorem (B. and Gramain)

Let p be an odd prime. Suppose $\mathfrak{S}_n$ has a p-basic set satisfying

1. $\chi_\lambda \in B$ if and only if $\chi_{\lambda^*} \in B$
2. $\chi_\lambda \in B$ with $\lambda = \lambda^*$ if and only if $[\overline{\lambda}]$ is p-regular.

Then $B_{\mathfrak{A}_n}$ is a p-basic set of $\mathfrak{A}_n$.

A candidate

- Let $\lambda \vdash n$. We associate to λ its p-core $\gamma(\lambda)$ and its p-quotient

$$q(\lambda) = \left(\lambda^{[1]}, \ldots, \lambda^{[p]}\right).$$

Note that the construction of the p-quotient depends on a choice of origin.

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  Note that the construction of the p-quotient depends on a choice of origin.
- With a fixed origin, the map

$$\lambda \in \mathcal{P}_n \mapsto (q(\lambda), \gamma(\lambda))$$

  is a bijection.

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Problem 2

Let $\lambda = \lambda^*$. Find conditions on $q(\lambda)$ such that $[\bar{\lambda}]$ is p-regular.

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Lemma

There is a choice of origin such that, if $q(\lambda) = (\lambda^{[1]}, \ldots, \lambda^{[p]})$ denotes the p-quotient of λ, then

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In particular, for $\lambda = \lambda^*$ and p odd:

$$q(\lambda) = \left(\lambda^{[1]}, \dots, \lambda^{[(p-1)/2]}, \lambda^{[(p+1)/2)]}, \lambda^{[(p-1)/2)]^*}, \dots, \lambda^{[1]^*}\right),$$

with $\lambda^{[(p+1)/2)]^*} = \lambda^{[(p+1)/2)]}$.

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Lemma

For an odd prime p and with the same choice of origin as above, if $\lambda = \lambda^*$, then $\overline{\lambda}$ is p-regular if and only if

$$\lambda^{[(p+1)/2]} = \emptyset.$$

Result

Theorem (B. and Gramain)

Let p be an odd prime and

$$\Lambda = \{(*, \ldots, *, \emptyset, * \ldots, *)\}.$$

Put

$$B_\Lambda = \{\chi_\lambda \mid q(\lambda) \in \Lambda\}.$$

Then B_Λ is a p-basic set of $\mathfrak{S}_n$, such that (1) and (2) are satisfed.

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It follows:

Corollary

For every odd prime p, $\mathfrak{A}_n$ has a p-basic set.

Generalized blocks

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- The equivalence classes are the $\mathcal{C}$-blocks of G.

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- Let $\mathcal{C} \subseteq \mathrm{Cl}(G)$.
- Put
$$\langle \chi, \chi' \rangle_{\mathcal{C}} = \langle \chi^{\mathcal{C}}, \chi'^{\mathcal{C}} \rangle_G,$$
with $\chi^{\mathcal{C}}(c) = \begin{cases} \chi(c) & \text{if c} \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}.$
- The transitive closure of the relation $\mathcal{R}$ defined by
$$\chi \,\mathcal{R}\, \chi' \Leftrightarrow \langle \chi, \chi' \rangle_{\mathcal{C}} \neq 0,$$
is an equivalence relation.
- The equivalence classes are the $\mathcal{C}$-blocks of G.
- Remark: $\mathcal{C}^{p\text{-reg}}$-block are the usual p-blocks in p-modular theory.

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Proposition

G has a $\mathcal{C}$-basic set if and only if every $\mathcal{C}$-block b of G has a $\mathcal{C}$-basic set.

Perfect isometries and $\mathcal{C}$-basic set

- Let $(b, \mathcal{C})$ and $(b', \mathcal{C}')$ be union of blocks of $b \subseteq \mathrm{Irr}(G)$ and $b' \subseteq \mathrm{Irr}(G')$, respectively.

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- $\Phi : b \to b'$ is a generalized perfect isometry if, for every $\chi \in b$, there is a sign $\eta(\chi) = \pm 1$ such that

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Proposition

Let $\Phi : (b, \mathcal{C}) \to (b', \mathcal{C}')$ be a perfect isometry. Then B is a $\mathcal{C}$-basic set of b if and only if $\Phi(B)$ is a $\mathcal{C}'$-basic set of b'.

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- Let b be a p-Block of $\mathfrak{S}_n$. Define the p-weight of b by $w = w_\lambda$ for every $\lambda \in b$.

A generalized perfect isometry

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- One can parametrize $\mathrm{Irr}(G_w)$ and $\mathrm{Cl}(G_w)$ with Γ_w.

$$\Gamma_w \to \mathrm{Irr}(G_w), \alpha \mapsto \phi_\alpha.$$

$$\Gamma_w \to \mathrm{Cl}(G_w), \alpha \mapsto [\alpha].$$

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Theorem

For an odd prime p and b a p-Block of $\mathfrak{S}_n$ of p-weight w, the map

$$\Phi : (b, p\text{-reg}) \longrightarrow (G_w, \mathcal{C}_\emptyset), \chi_\lambda \mapsto \phi_{\widetilde{q(\lambda)}}$$

is a generalized perfect isometry.

Simplification of the problem

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- Our problem is then equivalent to prove that

$$B_\emptyset = \{\phi_\alpha \,|\, \alpha \in \Lambda_w\}$$

is a $\mathcal{C}_\emptyset$-basic set of G_w.

A $\mathcal{C}_\emptyset$-basic set of G_w

- Note that
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- The map

$$[(\alpha_1, \dots, \alpha_{p-1})] \mapsto [(\alpha_1, \dots, \alpha_{p-1}, \emptyset)]$$

is a bijection between the classes of $\mathbb{Z}_{p-1} \wr \mathfrak{S}_w$ and the set of classes $\mathcal{C}_\emptyset$.

A $\mathcal{C}_\emptyset$-basic set of G_w

- For $\phi_\alpha \in B_\emptyset$, one has

$$\mathrm{Res}^{G_w}_{\mathcal{C}_\emptyset}(\phi_\alpha) \in \mathrm{Irr}(\mathbb{Z}_{p-1} \wr \mathfrak{S}_w).$$

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- Then $B_\emptyset$ is a $\mathcal{C}_\emptyset$-basic set of G_w.

Consequences

- Let D be the p-decomposition matrix of $\mathfrak{S}_n$.

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Lemma

The sign character ε acts on rows and columns of D_Λ. Moreover, these two operations are equivalent.

- Note that there is the same number of fixed rows and fixed columns.
- Denote by D_n the submatrix of D_Λ whose entries are all the entries of D_Λ lying at the intersection of an ε-stable row and an ε-stable column.

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Theorem

Suppose that we know

- The decomposition matrix of $\mathfrak{S}_n$.
- The submatrix $D_{\mathfrak{A}_n,n}$.

Then we can construct the decomposition matrix of $\mathfrak{A}_n$.

Questions

- Use the basic set B_Λ in order to parametrize the simple module of $\mathfrak{A}_n$ in characteristic p.

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- Connection with the Mullineux map.